

# Classification of affine matrix means

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**Abstract** In this article we find all possible matrix means which are points of geodesics of affinely connected manifolds. We characterize certain properties of these manifolds, decide whether they are metrizable or not. We show that the matrix means that occur in this form are exactly the matrix power means. We show that the several variable formulas of power means are the unique solutions of certain Karcher equations corresponding to affinely connected geometric structures. At the same time we construct weighted matrix means corresponding to a matrix mean and we show that the representing operator monotone functions of the weighted means form a one parameter family of functions that has a semigroup structure in the sense of Loewner. These one parameter families are induced by operator monotone functions that behave similarly as logarithm maps of affinely connected manifolds. We show that the holomorphicity of the members of these one-parameter families is controlled by the distribution of the branch points of the corresponding logarithm map.

**Keywords** operator monotone function · starlike function · matrix mean · symmetric space · affine connection

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## 1 Introduction

The open convex cone of positive definite  $n$ -by- $n$  matrices, denoted by  $P(n, \mathbb{C})$ , as a smooth manifold, carries some important Riemannian structures which are, in fact, symmetric spaces. For example consider the classical Riemannian symmetric space  $P(n, \mathbb{C}) \cong GL(n, \mathbb{C})/U(n, \mathbb{C})$  which plays an important role in several constructions [9]. This symmetric space is nonpositively curved, hence a unique minimizing geodesic between any two points exists. The midpoint operation on

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this space, which is defined as taking middle points of the geodesic connecting two points, is the geometric mean of two positive definite matrices [6].

The other two symmetric spaces which are well known are Euclidean spaces. One of them is just the vector space of squared complex matrices. The subset  $P(n, \mathbb{C})$  therefore inherits its Euclidean structure and the midpoint operation is the arithmetic mean. The third case is the harmonic mean, the corresponding space is isometric to the previous one.

These affinely connected manifold structures are important and widely used in several cases. For instance the extension of the two variable geometric mean to several variables are heavily based on the Riemannian structure of the corresponding space [1, 28, 25]. The multivariable geometric mean or Karcher mean is defined as the center of mass

$$A(A_1, \dots, A_n) = \arg \min_{X \in P(n, \mathbb{C})} \sum_{i=1}^n \delta^2(X, A_i). \quad (1)$$

on the Riemannian manifold  $P(n, \mathbb{C})$  endowed with the trace metric

$$d(A, B) = \sqrt{\text{Tr} \log(A^{-1}B)}.$$

This mean and the geometric structure corresponding to it is also used in practical applications when one considers averaging of symmetric tensors [2, 11, 21, 3, 4]. The Karcher mean  $A(A_1, \dots, A_n)$  is also the unique positive definite matrix where the gradient of the function in the minimization problem (1) vanishes

$$\sum_{i=1}^n \log(X^{-1}A_i) = 0.$$

This equation is called the Karcher equation. Recently Lim and Pálfi [26] found a one parameter family of multivariable matrix means called the matrix power means which are defined as the unique positive definite solution of the equation

$$X = \frac{1}{n} \sum_{i=1}^n G_t(X, A_i) \quad (2)$$

where  $G_t(A, B) = A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^t A^{1/2}$  is the weighted geometric mean of  $A, B \in P(n, \mathbb{C})$ . The attractive property of this family is that as the defining parameter  $t \rightarrow 0$ , the matrix power means converge to the Karcher mean. In this paper in Section 7 we will show that the matrix power means are actually unique solutions of Karcher equations

$$\sum_{i=1}^n \log_X(A_i) = 0$$

corresponding to affine connections given on the manifold  $P(n, \mathbb{C})$ . In other words the two-variable matrix power means give back the geodesic lines corresponding to these affine connections. Moreover in Section 8 we find all possible two-variable matrix means which occur such a way, in other words as affine matrix means,

which question was raised in [32]. It turns out that these means are exactly the matrix power means. We prove also that the corresponding affine connections are

$$\nabla_{X_p} Y_p = DY[p][X_p] - \frac{\kappa}{2} \left( X_p p^{-1} Y_p + Y_p p^{-1} X_p \right),$$

where  $0 \leq \kappa \leq 2$  and the tangent space is  $H(n, \mathbb{C})$ , the space of  $n$ -by- $n$  hermitian matrices, at every point  $p \in P(n, \mathbb{C})$ . These connections appear earlier when we construct them as prototypes of invariant affine connections in Section 6. In Section 9 among other results we show that these affine connections are non-metric in general, i.e. there exist no other Riemannian structures as in the case of the Karcher mean (1). In order to achieve this we investigate the holonomy groups and other properties of these affine connections.

During the above process we give some geometric construction which can be used over a general affinely connected space to reconstruct the logarithm (hence also the exponential) map of the corresponding affine connection from the midpoint operation  $m(p, q) = \exp_p(1/2 \log_p(q))$  on the manifold as

$$\log_p(q) = \lim_{n \rightarrow \infty} \frac{m(p, q)^{\circ n} - p}{\frac{1}{2^n}}$$

where we use the notation  $m(p, q)^{\circ n} \equiv m\left(p, m(p, q)^{\circ(n-1)}\right)$  and  $\log_p(q)$  is the logarithm map. This is discussed in Section 3. We apply an analogue of such a process to arbitrary two-variable matrix means in Section 4 and we obtain a corresponding logarithm map

$$\log_A(B) = A^{1/2} \log_I \left( A^{-1/2} B A^{-1/2} \right) A^{1/2}$$

to the matrix mean, where  $\log_I(x)$  is an operator monotone function. We show that  $\log_A(B)$ , hence  $\log_I(x)$  directly induce a one parameter family of operator monotone functions which represent matrix means that can be thought of the weighted counterparts of the original mean. Then in Section 5 we show that taking directly such operator monotone functions  $\log_I(x)$  that can be prototype of logarithm maps, we obtain similar one parameter families of matrix means. We show that in both cases these families are Loewner semigroups of Pick functions which itself has a classical and rich theory [12, 13, 17, 18, 19, 20]. We also show that the further extendability to greater parameter values of the one parameter family depends on the distribution of the branch points of the corresponding logarithm map. We relate this extendability property to functional equations over the upper complex half-plane of the form

$$\log_I(f_t(z)) = t \log_I(z), \quad (3)$$

where  $f_t(z)$  is the representing operator monotone function of the matrix mean and  $\log_I(z)$  is the corresponding logarithm map. We show that if  $\log_I(z)$  has no branch points in the upper half-plane, then the functional equation, hence the one parameter family  $f_t(z)$  is a Pick function, i.e. an operator monotone function for all  $t \in [0, 1]$ .

## 2 Matrix means and some constructions

Let us recall the family of matrix means [24]:

**Definition 1** A two-variable function  $M: P(n, \mathbb{C}) \times P(n, \mathbb{C}) \mapsto P(n, \mathbb{C})$  is called a matrix mean if

- (i)  $M(I, I) = I$  where  $I$  denotes the identity,
- (ii) if  $A \leq A'$  and  $B \leq B'$ , then  $M(A, B) \leq M(A', B')$ ,
- (iii)  $CM(A, B)C \leq M(CAC, CBC)$  for all hermitian  $C$ ,
- (iv) if  $A_n \downarrow A$  and  $B_n \downarrow B$  then  $M(A_n, B_n) \downarrow M(A, B)$ .

In property (ii), (iii), (iv) the partial order being used is the positive definite order, i.e.  $A \leq B$  if and only if  $B - A$  is positive semidefinite. An important consequence of these properties is [24] that every matrix mean can be uniquely represented by a normalized, operator monotone function  $f(t)$  in the following form

$$M(A, B) = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}. \quad (4)$$

This unique  $f(t)$  is said to be the representing function of the matrix mean  $M(A, B)$ . So actually matrix means are in one to one correspondence with normalized operator monotone functions, the above characterization provides an order-isomorphism between them. Normalization means that  $f(1) = 1$ . For symmetric means, i.e. for means  $M(A, B) = M(B, A)$ , we have  $f(t) = tf(1/t)$  which implies that  $f'(1) = 1/2$ . Operator monotone functions have strong continuity properties, namely all of them are analytic functions and can be analytically continued to the upper complex half-plane. This is the consequence of the integral characterization of an operator monotone function  $f(t)$ , which is given over the interval  $(0, \infty)$ :

$$f(t) = \alpha + \beta t + \int_0^\infty \left( \frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + t} \right) d\mu(\lambda), \quad (5)$$

where  $\alpha$  is a real number,  $\beta \geq 0$  and  $\mu$  is a unique positive measure on  $(0, \infty)$  such that

$$\int_0^\infty \frac{1}{\lambda^2 + 1} d\mu(\lambda) < \infty. \quad (6)$$

Actually the interval  $(0, \infty)$  may be changed to an arbitrary  $(a, b)$ , in this case the integral is transformed to this interval accordingly. These are the consequences of the theory of Loewner, an introduction to the theory can be found in Chapter V [5]. We will use this integral characterization at several points in the article in order to show that certain functions are analytic.

In this article we are interested in finding all possible symmetric matrix means which are also geodesic midpoint operations on smooth manifolds. Or more generally those matrix means that are arbitrary dividing points of geodesics. We will call such a matrix mean affine [32]:

**Definition 2 (Affine matrix mean)** An affine matrix mean  $M : W^2 \mapsto W$  is a matrix mean which is also a point of an arc-length parametrized geodesic on a smooth manifold  $W \supseteq P(n, \mathbb{C})$  equipped with an affine connection  $\nabla$ . I.e.  $M(A, B) = \exp_A(t \log_A(B))$  for a fixed  $t \in (0, 1)$  and for all  $A, B \in P(n, \mathbb{C})$ ,  $B$  is assumed to be in the injectivity radius of the exponential map  $\exp_A(x)$  of the connection  $\nabla$  given at the point  $A$ . The mapping  $\log_A(x)$  is just the inverse of the exponential map at the point  $A \in W$ .

We can make some basic observations about affine matrix means. First of all note, that by (4) we have that  $f(X) = M(I, X)$ , so if  $M(A, B)$  is an affine matrix mean, then  $f(X)$  is some point of a geodesic connecting  $X$  and  $I$ . Also on a smooth manifold with an affine connection if we differentiate the exponential map  $\exp_p(X)$  at  $p$ , then we get  $d\exp_p = I_p$ , where  $I_p$  is the identity transformation of the tangent space at  $p$  [15]. Therefore if we differentiate its inverse, the logarithm map  $\log_p(q)$  we also get  $d\log_p = I_p$  at  $p$ . So if we combine this with the chain rule we get that the differential of the mapping  $M(p, q) = \exp_p(t\log_p(q))$  is  $dM(p, \cdot) = tI_p$ .

Now if we apply the above argument to an affine matrix mean  $M(A, B)$  we get the following result.

**Proposition 1** *Let  $M(A, B) := \exp_A(t\log_A(B))$  be an affine matrix mean. Then  $f'(1) = t$ .*

*Proof* Since  $P(n, \mathbb{C})$  is diffeomorphically embedded in  $H(n, \mathbb{C})$ , therefore we can differentiate the map  $M(A, B) = \exp_A(t\log_A(B))$  using the vector space structure of  $H(n, \mathbb{C})$ , i.e. calculate the Fréchet differential which we denote for an arbitrary differentiable function  $g$  by

$$Dg[X][Y] = \lim_{s \rightarrow 0} \frac{g(X + Ys) - g(X)}{s} \quad (7)$$

at the matrix  $X$  in the direction of the matrix  $Y$ . So by (4) for all  $H \in H(n, \mathbb{C})$  we have

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{M(A, A + Hs) - M(A, A)}{s} \\ &= \lim_{s \rightarrow 0} A^{1/2} \frac{M(I, I + A^{-1/2}HA^{-1/2}s) - M(I, I)}{s} A^{1/2} = \\ &= A^{1/2} \lim_{s \rightarrow 0} \frac{f(I + A^{-1/2}HA^{-1/2}s) - f(I)}{s} A^{1/2} = A^{1/2} Df[I][A^{-1/2}HA^{-1/2}] A^{1/2}. \end{aligned}$$

Since  $f$  is an operator monotone function on  $(0, \infty)$ , it admits an integral characterization (5), so it can be analytically continued to the upper half-plane through the interval  $(0, \infty)$ . Therefore we may differentiate a power series representation of  $f$ , that uniformly converges on an open interval which contains 1, so then we get that  $Df[I][K] = Df[I][I]K = f'(1)K$  for all  $K \in H(n, \mathbb{C})$ . Combining this with the above we get that

$$\lim_{s \rightarrow 0} \frac{M(A, A + Hs) - M(A, A)}{s} = A^{1/2} \left( Df[I][I] A^{-1/2} H A^{-1/2} \right) A^{1/2} = f'(1)H.$$

Since  $H$  was arbitrary this yields that  $t = f'(1)$ , because  $dM(p, \cdot) = tI_p$  and also the tangent space of  $P(n, \mathbb{C})$  at every point can be identified by  $H(n, \mathbb{C})$ .  $\square$

By the preceding proposition we shall focus on matrix means represented by operator monotone functions  $f$  on  $(0, \infty)$  such that  $f'(1) \in (0, 1)$ . We will use the notation  $\mathfrak{P}(t)$  to denote the set of all operator monotone functions  $f$  on  $(0, \infty)$  such that  $f(x) > 0$  for all  $x \in (0, \infty)$  and  $f(1) = 1, f'(1) = t$ . We can find the minimal and maximal elements of  $\mathfrak{P}(t)$  for all  $t \in (0, 1)$  easily.

**Lemma 1** For all  $f(x) \in \mathfrak{P}(t)$  we have

$$\left((1-t) + tx^{-1}\right)^{-1} \leq f(x) \leq (1-t) + tx. \quad (8)$$

*Proof* Since every operator monotone function is operator concave, see Chapter V [5], therefore we must have  $f(x) \leq (1-t) + tx$  by concavity and the normalization conditions on elements of  $\mathfrak{P}(t)$ . Since the map  $x^{-1}$  is order reversing on positive matrices, we have that if  $f(x) \in \mathfrak{P}(t)$  then also  $f(x^{-1})^{-1} \in \mathfrak{P}(t)$ . So again by concavity

$$\begin{aligned} f(x^{-1})^{-1} &\leq (1-t) + tx \\ f(x^{-1}) &\geq ((1-t) + tx)^{-1} \\ f(x) &\geq \left((1-t) + tx^{-1}\right)^{-1}. \end{aligned}$$

□

Since  $\left((1-t) + tx^{-1}\right)^{-1}$  and  $(1-t) + tx$  are operator monotone we see that they are the minimal and maximal elements of  $\mathfrak{P}(t)$  respectively, and also they are the representing functions of the weighted harmonic and arithmetic means. This already gives us that the minimal and maximal affine matrix means are the weighted harmonic and arithmetic means respectively, so if  $M(A, B)$  is an affine matrix mean, then

$$\left[(1-t)A^{-1} + tB^{-1}\right]^{-1} \leq M(A, B) \leq (1-t)A + tB. \quad (9)$$

In general by the previous Lemma 1 the above inequality is true for all  $M(A, B)$  matrix means with representing operator monotone function  $f$  for which we have  $f'(1) = t$ . In this sense  $\mathfrak{P}(t)$  characterizes weighted matrix means. If we take this as the definition of weighted matrix means, one can compare it with the definition of weighted matrix means given in [32].

**Lemma 2** All  $f(x) \in \mathfrak{P}(t)$  for  $t \in (0, 1)$  has only one fixed point in  $(0, \infty)$  which is 1 and 1 is an attractive fixed point on  $(0, \infty)$ .

*Proof* By the definition of  $\mathfrak{P}(t)$  for all members  $f(x)$  of this set  $f(1) = 1$ , so 1 is indeed a fixed point. By the preceding Lemma 1 we have

$$\left((1-t) + tx^{-1}\right)^{-1} \leq f(x) \leq (1-t) + tx.$$

Therefore for all  $x > 1$  we have  $f(x) < x$ , i.e.  $f(x)$  has no fixed point in  $(1, \infty)$ . Similarly for all  $x \in (0, 1)$  we have

$$x < \left((1-t) + tx^{-1}\right)^{-1} \leq f(x),$$

therefore  $f(x)$  cannot have a fixed point in  $(0, 1)$  as well.

Now the attractivity of the fixed point follows from the fact that  $f(x)$  is monotonically increasing positive and concave on  $(0, \infty)$  by operator monotonicity. Concavity implies that  $f''(x) \leq 0$  for all  $x \in (0, \infty)$ . Also its derivative  $f'(1) = t \in (0, 1)$

at the fixed point 1, so by Banach's fixed point theorem this fixed point is attractive on  $(\epsilon, \infty)$ , where  $\epsilon < 1$  is such that the derivative  $f'(\epsilon) = 1$ . On  $(0, \epsilon)$  the function  $(1 - t) + tx \geq f(x) > x$  so its subsequent iterates form an increasing sequence of functions. I.e. if we start an iteration with  $x_0 \in (0, \epsilon)$ , then after finitely many iterations by  $f(x)$ ,  $x_n = f(x_{n-1})$  will be in the interval  $(\epsilon, 1)$ . From there convergence to 1 follows again from Banach's fixed point theorem.  $\square$

In order to advance further in the understanding of affine matrix means, we should be able to grasp more geometrical structure related to the affinely connected manifolds corresponding to affine matrix means. In the next section we will study the general situation of affinely connected manifolds given with a geodesic dividing point operation. We will see that in this case we can reconstruct the exponential map and its inverse, the logarithm map from the geodesic dividing point operation.

### 3 The reconstruction of the exponential map

In this section we reconstruct the exponential map of an arbitrary affinely connected differentiable manifold based first on its midpoint map. Without loss of generality we fix a base point  $p$  as the starting point of the geodesics. The basics of the exponential map of a manifold can be found for example in Chapter I. paragraph 6 [15].

**Theorem 1** *Let  $M$  be an affinely connected smooth manifold diffeomorphically embedded into a vector space  $V$ . Suppose that the midpoint map  $m(p, q) = \exp_p(1/2 \log_p(q))$  is known in every normal neighborhood where the exponential map  $\exp_p(X)$  is a diffeomorphism. Then in these normal neighborhoods the inverse of the exponential map  $\log_p(q)$  can be fully reconstructed from the midpoint map in the form*

$$\log_p(q) = \lim_{n \rightarrow \infty} \frac{m(p, q)^{\circ n} - p}{\frac{1}{2^n}}, \quad (10)$$

where we use the notation  $m(p, q)^{\circ n} \equiv m(p, m(p, q)^{\circ(n-1)})$ .

*Proof* We will use some basic properties of the differential of the exponential map to construct the inverse of it, the logarithm map. Since in small enough normal neighborhoods the exponential map is a diffeomorphism, it can be given as the inverse of the logarithm map  $\log_p(q)$ .

By the basic properties of the exponential map we have

$$\left. \frac{\partial \exp_p(Xt)}{\partial t} \right|_{t=0} = \lim_{t \rightarrow 0} \frac{\exp_p(Xt) - p}{t} = X,$$

where  $X \in T_p M$ . Here we used the fact that we have an embedding into a vector space. Suppose  $\exp_p(X) = q$  is in the normal neighborhood. We are going to provide the limit on the right hand side of the above equation. The limit clearly exists in the normal neighborhood so

$$\lim_{t \rightarrow 0} \frac{\exp_p(Xt) - p}{t} = \lim_{n \rightarrow \infty} \frac{\exp_p(X \frac{1}{2^n}) - p}{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \frac{m(p, q)^{\circ n} - p}{\frac{1}{2^n}}.$$

Here we use the notation  $m(p, q)^{\circ n} \equiv m\left(p, m(p, q)^{\circ(n-1)}\right)$ . We are in a normal neighborhood so the exponential map has an inverse, the logarithm map, so the limit can be written as

$$X = \lim_{t \rightarrow 0} \frac{\exp_p(Xt) - p}{t} = \lim_{n \rightarrow \infty} \frac{m(p, q)^{\circ n} - p}{\frac{1}{2^n}} = \log_p(q).$$

□

In the above assertion we used the midpoint map to reconstruct the exponential map, but we can use arbitrary dividing point operation that yields a point, other than the ending points on the geodesic connecting two points in the normal neighborhood. This is summarized in the following proposition.

**Proposition 2** *Let  $M$  be an affinely connected smooth manifold diffeomorphically embedded into a vector space  $V$ . In every normal neighborhood  $N$  let  $\gamma_{a,b}(t)$  denote the geodesic connecting  $a, b \in N$  with parametrization  $\gamma_{a,b}(0) = a$  and  $\gamma_{a,b}(1) = b$ . Suppose that the map  $m(a, b)_{t_0} = \gamma_{a,b}(t_0) = \exp_p(t_0 \log_p(q))$  is known for a  $t_0 \in (0, 1)$  in every normal neighborhood  $N$  where the exponential map is a diffeomorphism and  $a, b \in N$ . Then in these normal neighborhoods the logarithm map can be fully reconstructed as*

$$\log_p(q) = \lim_{n \rightarrow \infty} \frac{m(p, q)_{t_0}^{\circ n} - p}{t_0^n},$$

with the notation  $m(p, q)_{t_0}^{\circ n} \equiv m\left(p, m(p, q)_{t_0}^{\circ(n-1)}\right)$ . We also obtain the exponential map by inverting  $\log_p(q)$ .

We are going to use this construction in the next sections to characterize affine matrix means.

#### 4 The exponential map of affine matrix means

Based on the idea of reconstruction given by Proposition 2 we are going to formally take the limits for matrix means in  $\mathfrak{P}(t)$ . The following result will show that if a matrix mean is affine then the exponential map of the corresponding smooth manifold has a special structure. We will use similarly the notation  $M(A, B)^{\circ n} = M\left(A, M(A, B)^{\circ(n-1)}\right)$  as before in the previous section.

**Theorem 2** *Let  $M(A, B)$  be a matrix mean with representing function  $f \in \mathfrak{P}(t)$ . Then*

$$\lim_{n \rightarrow \infty} \frac{M(A, B)^{\circ n} - A}{f'(1)^n} = A^{1/2} \log_I \left( A^{-1/2} B A^{-1/2} \right) A^{1/2} \quad (11)$$

where the limit exists and is uniform for all  $A, B \in P(n, \mathbb{C})$  and  $\log_I(x)$  is an operator monotone function which fulfills the functional equation

$$\log_I(f(x)) = f'(1) \log_I(x) \quad (12)$$

on the interval  $(0, \infty)$ .



*Proof* We will prove the convergence to a continuous function  $\log_I(t)$  in a more general setting. The operator monotonicity in the matrix mean case will be a particularization.

First of all note that by the repeated usage of (4) we can reduce the above problem to the right hand side of the following formula:

$$\frac{M(A, B)^{\circ n} - A}{f'(1)^n} = A^{1/2} \frac{f\left(A^{-1/2}BA^{-1/2}\right)^{\circ n} - I}{f'(1)^n} A^{1/2}.$$

From now on we will explicitly use the notation  $g(x)^{\circ n} = g\left(g(x)^{\circ(n-1)}\right)$  for arbitrary function  $g(t)$  where this notation is straightforward.

Due to the above formula it is enough to prove the assertion for a single operator monotone function  $f(x)$ . By operator monotonicity of  $f(x)$  this is just the special case of the problem considered for arbitrary concave, analytic functions  $f(x)$  given in the following form

$$\lim_{n \rightarrow \infty} \frac{f(X)^{\circ n} - I}{f'(1)^n}, \quad (13)$$

for  $X \in P(n, \mathbb{C})$ . As every operator monotone function which maps  $(0, \infty)$  to  $(0, \infty)$ , is analytic on  $(0, \infty)$  and has an analytic continuation to the complex upper half-plane across  $(0, \infty)$ , we can consider the functional calculus for hermitian matrices in the above equations. Therefore we can further reduce the problem to the set of the positive reals by diagonalizing  $X$  and considering the convergence for every distinct diagonal element separately. For an extensive study on operator monotone functions one may refer to Chapter V in [5].

Without loss of generality we may shift the function  $f(x)$  by 1 so it is enough to show the assertion for

$$\lim_{n \rightarrow \infty} \frac{g(x)^{\circ n}}{g'(0)^n},$$

where  $g(x) = f(x+1) - 1$  and so  $g(x)^{\circ n} = f(x+1)^{\circ n} - 1$ . From now on we will be considering the shifted problem. At this point we must emphasize the fact that the function  $g$  must have 0 as an attractive and only fixed point on the interval of interest  $(-1, \infty)$ . In the unshifted case this is equivalent to  $f$  having 1 as the only attractive fixed point on the interval  $(0, \infty)$ , which is the case by Lemma 2. So we can also assume that  $0 < g'(0) < 1$ . The rest of the argument will be based on the claim that the above limit of analytic functions of the form  $g(x)^{\circ n}/g'(0)^n$  is uniform Cauchy therefore the limit function exists and is continuous.

First of all we have 0 as the attractive and only fixed point of  $g$ , so for arbitrary  $x \in (-1, \infty)$  the sequence  $x_n = g(x)^{\circ n}$  converges to 0. We have  $g(0) = 0$  and by the mean value theorem we have

$$x_n = g(x)^{\circ n} = g'(t_n)g(x)^{\circ(n-1)} = \prod_{i=1}^n g'(t_i)x,$$

where  $t_i \in [0, g(x)^{\circ(i-1)}]$  if  $x \geq 0$  or  $t_i \in [g(x)^{\circ(i-1)}, 0]$  if  $x < 0$ , since  $g$  is a concave function on  $(-1, \infty)$ . As  $x_n \rightarrow 0$  for arbitrary  $x$  we have  $g'(t_i) \rightarrow g'(0)$ . Now we have to obtain a suitable upper bound on

$$\left| \frac{g(x)^{\circ n}}{g'(0)^n} - \frac{g(x)^{\circ m}}{g'(0)^m} \right|. \quad (14)$$

We argue as follows

$$\begin{aligned} \left| \frac{g(x)^{\circ n}}{g'(0)^n} - \frac{g(x)^{\circ m}}{g'(0)^m} \right| &= \frac{|g(x)^{\circ n} - g'(0)^{n-m} g(x)^{\circ m}|}{g'(0)^n} \leq \\ &\leq \frac{|\prod_{i=m+1}^n g'(t_i) - g'(0)^{n-m}| |\prod_{i=1}^m g'(t_i)|}{g'(0)^n} |x| = \\ &= \left| \prod_{i=m+1}^n \frac{g'(t_i)}{g'(0)} - 1 \right| \left| \prod_{i=1}^m \frac{g'(t_i)}{g'(0)} \right| |x|. \end{aligned}$$

Now uniform convergence follows if  $|\prod_{i=1}^{\infty} g'(t_i)/g'(0)| < \infty$  because then the tail  $\prod_{i=m+1}^{\infty} g'(t_i)/g'(0) \rightarrow 1$  so (14) can be arbitrarily small on any closed interval in  $(-1, \infty)$  by choosing a uniform  $m$ . By the continuity of  $g'(t)$  and  $x_n \rightarrow 0$  we have  $g'(t_i) \rightarrow g'(0)$  and by assumption  $0 < g'(0) < 1$ , therefore there exists  $N$  and  $q$  such that for all  $i > N$  we have  $0 < g'(t_i) \leq q < 1$ . What follows here is that  $\exists K_1, K_2 < \infty$  such that  $|t_N| \leq K_1$  and  $|g''(t_i)| \leq K_2$  for all  $i > N$ . This yields the bound  $|t_i| \leq K_1 q^{i-N}$  for all  $i > N$ . Considering the Taylor expansion of  $g'$  around 0 we get

$$\frac{g'(t_i)}{g'(0)} = \frac{g'(0) + g''(t'_i)t_i}{g'(0)}$$

for  $0 < t'_i < t_i$ . What follows from this is that

$$\left| \prod_{i=N}^{\infty} \frac{g'(t_i)}{g'(0)} \right| \leq \prod_{i=N}^{\infty} \left( 1 + \frac{K_1 K_2}{g'(0)} q^{i-N} \right).$$

The infinite product on the right hand side converges because  $\sum_{j=0}^{\infty} \frac{K_1 K_2}{g'(0)} q^j$  converges hence  $|\prod_{i=1}^{\infty} g'(t_i)/g'(0)| < \infty$  for all  $x$  in the closed interval.

At this point we can easily establish the convergence for normalized operator monotone functions because they are concave functions by Theorem V.2.5 in [5], so  $f''(t) \leq 0$  and they have only one fixed point which is 1. The fact that the limit is operator monotone function in this case follows from the operator monotonicity of the generating  $f(t)$ .

The functional equation (12) is the consequence of the following:

$$\begin{aligned} \log_I(f(x)) &= \lim_{n \rightarrow \infty} \frac{f(f(x))^{\circ n} - 1}{f'(1)^n} = \lim_{n \rightarrow \infty} \frac{f(x)^{\circ(n+1)} - 1}{f'(1)^n} \\ &= \lim_{n \rightarrow \infty} f'(1) \frac{f(x)^{\circ(n+1)} - 1}{f'(1)^{n+1}} = f'(1) \log_I(x). \end{aligned}$$

□

Actually the above proof works for a larger class of functions than the family of normalized operator monotone functions. The limit in (13) exists and it is a continuous function if the twice differentiable function  $f(x)$  has 1 as the only attractive fixed point and the derivative  $-1 < f'(x) < 1$ . The next example shows how to calculate the limit function explicitly.

*Example 1* Consider the one-parameter family of functions

$$f_q(x) = [(1-t) + tx^q]^{1/q}$$

for  $t \in (0, 1)$ . These are in  $\mathfrak{P}(t)$  if and only if  $q \in [-1, 1]$ , because for other values of  $q$  the function is not operator monotone, see exercise 4.5.11 in [6]. It is easy to see that

$$f_q(x)^{\circ n} = \left[ t^n x^q + \sum_{k=0}^{n-1} t^k (1-t) \right]^{1/q} = [t^n x^q - t^n + 1]^{1/q}.$$

In this case we can easily calculate the limit function  $\log_{I, f_q}(x)$  by turning the limit into a derivative:

$$\begin{aligned} \log_{I, f_q}(x) &= \lim_{n \rightarrow \infty} \frac{(t^n x^q - t^n + 1)^{1/q} - 1}{t^n} = \lim_{s \rightarrow 0} \frac{(sx^q - s + 1)^{1/q} - 1}{s} \\ &= \frac{\partial}{\partial s} (sx^q - s + 1)^{1/q} \Big|_{s=0} = \frac{x^q - 1}{q}. \end{aligned}$$

The limit functions indeed are operator monotone again if and only if  $q \in [-1, 1]$ . This family has a singularity at  $q = 0$  but it is easy to verify that it is a removable singularity, so in fact we have

$$\begin{aligned} f_0(x) &= x^t \\ \log_{I, f_0}(x) &= \log(x), \end{aligned}$$

where  $\log(x)$  and  $x^t$  are also well known to be operator monotone. Particularly  $x^t$  as a representing function corresponds to the weighted geometric mean.

**Proposition 3** *The limit function  $\log_I(x)$  in Theorem 2 satisfies the following:*

- (i)  $\log_I(x)$  maps  $P(n, \mathbb{C})$  to  $H(n, \mathbb{C})$  injectively,
- (ii)  $1 - x^{-1} \leq \log_I(x) \leq x - 1$  for all  $x > 0$ ,
- (iii) If  $\log_{I, f}(x)$  and  $\log_{I, g}(x)$  are the corresponding limit functions for  $f, g \in \mathfrak{P}(t)$  such that  $f(x) \leq g(x)$  for all  $x > 0$ , then  $\log_{I, f}(x) \leq \log_{I, g}(x)$  for all  $x > 0$ ,
- (iv)  $\log_I(1) = 0$  and  $\log'_I(1) = 1$ .

*Proof* (iii): Since  $f(x) \leq g(x)$  by monotonicity we have  $f(x)^{\circ n} \leq g(x)^{\circ n}$ . From this it follows that

$$\frac{f(x)^{\circ n} - 1}{f'(1)^n} \leq \frac{g(x)^{\circ n} - 1}{g'(1)^n},$$

and the inequality is also preserved in the limit.

(ii): By Lemma 1 we have

$$\left( (1-t) + tx^{-1} \right)^{-1} \leq f(x) \leq (1-t) + tx$$

where on the left hand side we have the function  $f_{-1}(x)$  and on the right hand side we have  $f_1(x)$  from Example 1. In Example 1 we calculated the corresponding limit functions, so these combined with the previous property (iii) proves property (ii).

(i): By property (ii) it follows that  $\log_I(x)$  is nonconstant on  $(0, \infty)$ . Also  $\log_I(x)$  is operator monotone there, so it is strictly concave, therefore injective

and real valued. This combined with the functional calculus for matrix functions proves the property.

(iv):  $\log_I(1) = 0$  follows from (ii). Using this and (ii) again we have

$$\frac{1 - (1+h)^{-1}}{h} \leq \frac{\log_I(1+h) - \log_I(1)}{h} \leq \frac{(1+h) - 1}{h}.$$

Taking the limit  $h \rightarrow 0$  we get derivatives on the left and right hand sides are 1, so also  $\log'_I(1) = 1$ .  $\square$

Since  $\log_I(x)$  is operator monotone on  $(0, \infty)$ , it is also analytic there, so it has an analytic inverse  $\exp_I(x)$  by Lagrange's inversion theorem, since its derivative is nonzero due to Proposition 3. It is also easy to see that  $\exp'_I(0) = 1$  and  $\exp_I(0) = 1$ . By these considerations we have just arrived at the following

**Proposition 4** *Let  $f \in \mathfrak{P}(t)$ . Then*

$$f(x) = \exp_I(f'(1) \log_I(x)), \quad (15)$$

where  $\log_I(x)$  is the unique solution of the functional equation (12) in the class of functions that are continuous, invertible on  $(0, \infty)$ , vanish at 1 and have derivative 1 at 1.

*Proof* The first part of the assertion follows from the analyticity and invertibility of  $\log_I(x)$  on  $(0, \infty)$ . For the second part note that if  $\log_I(x)$  is an invertible solution of the functional equation (12) and also  $\log_I(1) = 0$  and  $\log'_I(1) = 1$ , then its inverse  $\exp_I(x)$  exists,  $\exp_I(0) = 1$  and  $\exp'_I(0) = 1$ . Moreover

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(x)^{\circ n} - 1}{f'(1)^n} &= \lim_{n \rightarrow \infty} \frac{\exp_I(f'(1)^n \log_I(x)) - \exp_I(0)}{f'(1)^n} \\ &= \lim_{s \rightarrow 0} \frac{\exp_I(s \log_I(x)) - \exp_I(0)}{s} = \left. \frac{\partial}{\partial s} \exp_I(s \log_I(x)) \right|_{s=0} \\ &= \log_I(x). \end{aligned}$$

$\square$

The above propositions put some restrictions on the possible functions  $\log_I(x)$  that can occur as limits in Theorem 2. Therefore we will use the notation  $\mathfrak{L}$  to denote the set of operator monotone functions  $g(x)$  on  $(0, \infty)$  such that  $g(1) = 0$  and  $g'(1) = 1$ . By Proposition 4 it is clear, that for each  $f \in \mathfrak{P}(t)$  we have a unique corresponding  $\log_I(x)$  in  $\mathfrak{L}$ . We will further say that for an arbitrary  $f \in \mathfrak{P}(t)$  the corresponding solution  $\log_I(x)$  of the functional equation (12) is the logarithm map corresponding to  $f(x)$ , while its inverse  $\exp_I(x)$  is the exponential map corresponding to  $f(x)$ .

In the following section we will go the other way around and see whether the function

$$f(x) = \exp_I(t \log_I(x))$$

is in  $\mathfrak{P}(t)$  for all  $\log_I \in \mathfrak{L}$  and  $t \in (0, 1)$ .

## 5 Representing functions induced by logarithm maps

In the previous section we established that for every  $f \in \mathfrak{P}(t)$  there exists a unique function  $\log_I \in \mathfrak{L}$  such that it fulfills the functional equation (12). In order to see whether an element  $\log_I \in \mathfrak{L}$  also induces a representing function  $f \in \mathfrak{P}(t)$  with the generalized functional equation

$$\log_I(f(x)) = t \log_I(x)$$

for all  $t \in (0, 1)$ , we must extend our investigations into the upper complex half-plane  $\mathbb{H}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ .

First of all let us recall Nevanlinna's representation [5] of holomorphic functions  $f : \mathbb{H}^+ \rightarrow \mathbb{H}^+$ . By Nevanlinna's theorem each such  $f$  can uniquely be written as

$$f(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \frac{\lambda z + 1}{\lambda - z} d\nu(\lambda), \quad (16)$$

where  $\alpha \in \mathbb{R}, \beta \geq 0$  and  $\nu$  is a positive measure with support in  $(-\infty, \infty)$ . It is well known that  $f$  can be extended to the lower half-plane  $\mathbb{H}^- = \{z \in \mathbb{C} : \Im z < 0\}$  as well by Schwarz reflection  $\overline{f(\bar{z})}$  for all  $z \in \mathbb{H}^-$ . Therefore also if this extension is by analytic continuation over an interval  $(a, b)$ , then  $\nu$  vanishes on the interval [5]. Similarly if  $\nu$  vanishes on a real interval, then  $f$  is holomorphic on the interval as well and can be analytically continued to the lower half-plane.

Representation (16) will be useful for studying functions in  $\mathfrak{L}$ . For example Nevanlinna's representation yields that all  $f \in \mathfrak{L}$  can be represented as

$$f(z) = \alpha + \beta z + \int_{-\infty}^0 \frac{\lambda z + 1}{\lambda - z} d\nu(\lambda), \quad (17)$$

where  $\alpha \in \mathbb{R}, \beta \geq 0$  and  $\nu$  is a positive measure with support in  $(-\infty, 0)$ . This is due to the required holomorphicity of  $f$  on  $(0, \infty)$ . Next let us find the maximal and minimal elements in  $\mathfrak{L}$ .

**Lemma 3** *For all  $f(x) \in \mathfrak{L}$  we have*

$$1 - x^{-1} \leq f(x) \leq x - 1. \quad (18)$$

*Proof* Since every operator monotone function is operator concave, therefore we must have  $f(x) \leq x - 1$  by concavity and the normalization conditions on elements of  $\mathfrak{L}$ . Since the map  $x^{-1}$  and  $-x$  is order reversing on hermitian matrices, we have that if  $f(x) \in \mathfrak{L}$  then also  $-f(x^{-1}) \in \mathfrak{L}$ . So again by concavity

$$\begin{aligned} -f(x^{-1}) &\leq x - 1 \\ f(x) &\geq 1 - x^{-1}. \end{aligned}$$

Clearly  $x - 1$  and  $1 - x^{-1}$  are also in  $\mathfrak{L}$ .  $\square$

At this point let us refer again to the functional equation (12) in the previous section. By the above considerations we can generalize (12) by analytic continuation.

**Proposition 5** *Let  $f \in \mathfrak{P}(t)$ . Then the function  $\log_I(x)$  given in Theorem 2 admits analytic continuation to  $\mathbb{H}^+$  and also to  $\mathbb{H}^-$  across  $(0, \infty)$  by reflection, moreover it fulfills the functional equation*

$$\log_I(f(z)) = f'(1) \log_I(z) \quad (19)$$

for all  $z \in \mathbb{C} \setminus (-\infty, 0]$ .

*Proof* Since analytic continuation of  $f(x)$  and  $\log_I(x)$  can be performed using the integral characterizations (5) and (17) respectively, we end up with holomorphic functions living on  $\mathbb{C} \setminus (-\infty, 0]$ . Since  $\log_I(z)$  is a holomorphic function, it has a meromorphic inverse  $\exp_I(z)$ . So we have

$$F(z) = \exp_I(f'(1) \log_I(z)),$$

a meromorphic function that is identical to  $f(z)$  everywhere on the domain  $(0, \infty)$ . Therefore by uniqueness of meromorphic and analytic continuation we must have  $F(z) = f(z)$  everywhere on the domain  $\mathbb{C} \setminus (-\infty, 0]$ .  $\square$

The above result tells us, that for a given  $\log_I \in \mathfrak{L}$  we should consider the generalized functional equation

$$\log_I(f_t(z)) = t \log_I(z) \quad (20)$$

to define a representing function  $f_t(z)$  for all  $t \in (0, 1)$  corresponding to  $\log_I(z)$  which was itself obtained by analytic continuation using representation (17). The obvious question that arises here is whether every  $\log_I(z)$  in  $\mathfrak{L}$  has a corresponding  $f_t \in \mathfrak{P}(t)$ ? We need the following:

**Definition 3 (Radial convexity)** Let  $S \subseteq \mathbb{C}$  be such that  $0 \in S$ . We will say that  $S$  is radially convex if and only if for all  $z \in S$  also  $tz \in S$  for all  $t \in [0, 1]$ .

**Proposition 6** *Let  $\log_I \in \mathfrak{L}$ . Then  $\log_I$  maps  $\mathbb{H}^+$  to a radially convex set in  $\mathbb{H}^+$ .*

*Proof* Using Nevanlinna's characterization (17) we have that  $\log_I$  is a convex combination of functions of the form

$$h_\lambda(z) = \frac{\lambda z + 1}{\lambda - z}.$$

If we have  $h_\lambda(z) = w$  for  $z \in \mathbb{H}^+$ , then after some calculations we get that

$$z = \lambda - (\lambda^2 + 1) \frac{\lambda + \bar{w}}{|\lambda + w|^2}$$

which means that

$$\Im z = \frac{\lambda^2 + 1}{|\lambda + w|^2} \Im w.$$

In other words if  $h_\lambda(z) = w \in \mathbb{H}^+$ , then for all  $s \in (0, 1)$  there exists  $z_s \in \mathbb{H}^+$  such that  $h_\lambda(z_s) = sw$ .

Now if we consider any convex combination of such functions  $h_\lambda(z)$ , the resulting function will still have a radially convex image of  $\mathbb{H}^+$ . The reason for this is that if  $x_i \in S_i \subseteq \mathbb{H}^+$  where  $S_i$  are radially convex sets, then  $sx_i \in S_i$  for all

$s \in (0, 1)$ . Therefore if  $S_i$  are the images of  $\mathbb{H}^+$  under the mappings  $K_i h_{\lambda_i}(z)$  for some  $K_i > 0$  and  $\lambda_i$ , then the image  $S$  of  $\mathbb{H}^+$  under the function that we get as the sum of the functions  $K_i h_{\lambda_i}(z)$ , is radially convex, since every element of it can be written as a sum of some  $x_i \in S_i$ . So we also have that the sum of  $s x_i$  is in  $S$  too by the convexity of each  $S_i$ . Therefore  $S$  must be radially convex.  $\square$

**Theorem 3** *Let  $\log_I \in \mathfrak{L}$ . Then  $f_t \in \mathfrak{P}(t)$  for all  $t \in (0, 1)$  if and only if  $\log_I(z)$  has no branch point in  $\mathbb{H}^+$ .*

*Proof* First of all since  $\log_I \in \mathfrak{L}$ , it follows that  $\log_I(x)$  is invertible on  $(0, \infty)$  because it is nonconstant monotone increasing there, also it is invertible in a neighborhood of  $(0, \infty)$  and its inverse  $\exp_I(z)$  is holomorphic in that neighborhood and  $f_t(x) \in (0, \infty)$  and meromorphic in  $\mathbb{H}^+$ .

Suppose that  $\log_I(z)$  has no branch point in  $\mathbb{H}^+$ . Then by the previous Proposition 6 it maps  $\mathbb{H}^+$  to a radially convex set. Since  $\log_I(z)$  has no branch point, it has a univalent holomorphic inverse  $\exp_I(z)$ , so

$$f_t(z) = \exp_I(t \log_I(z))$$

is a well defined holomorphic function on  $\mathbb{H}^+$ . Moreover  $f_t(z)$  is real valued over  $(0, \infty)$ . Since the image  $\log_I(\mathbb{H}^+)$  of  $\mathbb{H}^+$  under the map  $\log_I(z)$  is radially convex, we have that for any  $s \in \log_I(\mathbb{H}^+)$  also  $ts \in \log_I(\mathbb{H}^+)$ . Therefore  $t \log_I(\mathbb{H}^+) \subseteq \log_I(\mathbb{H}^+)$ , so also  $\exp_I(t \log_I(\mathbb{H}^+)) \subseteq \mathbb{H}^+$ .

Now for the only if part suppose on the contrary that  $\log_I$  has a branch point in  $\mathbb{H}^+$ . Then its inverse  $\exp_I$  has a pole at the image of the branch point under  $\log_I$  which means that  $f_t(z)$  is not holomorphic there, but this contradicts  $f_t \in \mathfrak{P}(t)$ .  $\square$

What happens if  $\log_I$  has a branch point in  $\mathbb{H}^+$ ? What can then be said about  $f_t(z)$ ?

**Proposition 7** *Let  $\log_I \in \mathfrak{L}$  be induced by an  $f_{t_0} \in \mathfrak{P}(t_0)$  using Proposition 4. Then  $f_t \in \mathfrak{P}(t)$  for all  $0 < t \leq t_0$ .*

*Proof* By Proposition 4 we have that there is no image of a branch point of  $\log_I$  in the domain  $t_0 \log_I(\mathbb{H}^+) \subseteq \mathbb{H}^+$ , otherwise  $f_{t_0}(z)$  would have a singularity in  $\mathbb{H}^+$ . But since for all  $0 < t \leq t_0$  we have that  $t \log_I(\mathbb{H}^+) \subseteq t_0 \log_I(\mathbb{H}^+)$ , therefore  $f_t(z)$  is singularity free as well.  $\square$

*Remark 1* In general one can assure that if for a given  $\log_I \in \mathfrak{L}$  with branch points  $t \log_I(\mathbb{H}^+)$  avoids the image of the branch points (of  $\log_I$ ) under  $\log_I$  in  $\mathbb{H}^+$ , then  $f_t \in \mathfrak{P}(t)$ .

According to Theorem 3 we need to find members of  $\mathfrak{L}$  without branch points. In other words we are looking for mappings that are univalent (schlicht) holomorphic functions on  $\mathbb{H}^+$  mapping  $\mathbb{H}^+$  into itself. Such mappings are characterized by FitzGerald in the classical article [12].

**Theorem 4 (FitzGerald)** *Suppose  $f(x)$  is a twice continuously differentiable, real-valued function with positive first derivative on  $(a, b)$ . Suppose the origin is in  $(a, b)$  and  $f(0) = 0$ . A necessary and sufficient condition that  $f$  can be continued to be a*

univalent analytic function of  $\mathbb{H}^+$  onto a subset of itself that is radially convex with respect to the origin is that the function

$$\eta(x) = -\frac{f(x)}{f'(x)}$$

be conditionally positive definite, i.e.

$$\int_a^b \int_a^b \phi(s) \frac{\eta(s) - \eta(t)}{s - t} \phi(t) ds dt \geq 0$$

for all real continuous  $\phi$  having compact support in  $(a, b)$  and satisfying  $\int_a^b \phi(s) ds = 0$ , where  $\frac{\eta(s) - \eta(t)}{s - t}$  is identified with  $\eta'(s)$ .

Summarizing the results in the previous sections from the point of view of affine matrix means we arrive at the following

**Proposition 8** *If a matrix mean  $M(A, B)$  is affine, then the exponential map and its inverse, the logarithm map are of the following forms*

$$\begin{aligned} \exp_p(X) &= p^{1/2} \exp_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} \\ \log_p(X) &= p^{1/2} \log_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} \end{aligned} \quad (21)$$

for  $p \in P(n, \mathbb{C})$ , where  $\exp_I(X)$  and  $\log_I(X)$  are analytic functions such that  $\exp_I : H(n, \mathbb{C}) \mapsto P(n, \mathbb{C})$  and  $\log_I(X)$  is its inverse and  $\log'_I(I) = I, \exp'_I(0) = I, \log_I(I) = 0, \exp_I(0) = I$ .

Note that by Weierstrass's approximation theorem we also have

$$\begin{aligned} p^{1/2} \exp_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} &= p \exp_I \left( p^{-1} X \right) \\ p^{1/2} \log_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} &= p \log_I \left( p^{-1} X \right). \end{aligned} \quad (22)$$

In some cases, to ensure easier reading, similarly as in the above formulas, we will denote matrices with uppercase letters which are elements of some tangent space, while at the same time we will use lowercase letters for denoting matrices which are points of a differentiable manifold.

## 6 Construction of an invariant affine connection

Let us recall the classical symmetric space  $GL(n, \mathbb{C})/U(n, \mathbb{C})$ , the cone of positive definite  $n \times n$  matrices  $P(n, \mathbb{C})$  [9]. This is a Lie group and the  $K = U(n, \mathbb{C})$  isotropy group invariant inner product at the identity  $I$  is  $\langle U, V \rangle = \text{Tr} \{UV\}$ . The tangent space, considering the Cartan decomposition of the Lie algebra, is the space of Hermitian matrices  $H(n, \mathbb{C})$ . The action of the isometry group  $GL(n, \mathbb{C})$  on this manifold is  $g(o) = gog^*$  and acting with left translations we can transport the inner product to any point  $p$  on this manifold and we get the Riemannian



metric  $\langle U, V \rangle_p = \text{Tr} \{ p^{-1} U p^{-1} V \}$ . The exponential map is just the ordinary matrix exponential at the identity. The left invariant affine connection is

$$\nabla_{X_p} Y_p = DY[p][X_p] - \frac{1}{2} \left( X_p p^{-1} Y_p + Y_p p^{-1} X_p \right), \quad (23)$$

here  $DY[p][X_p]$  denotes the Fréchet-differential of  $Y$  at the point  $p$  in the direction  $X_p$ . A well known property of this metric is that the midpoint map of the space  $m(p, q) = \exp_p(1/2 \log_p(q))$  is just the geometric mean of two positive matrices

$$G(A, B) = A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^{1/2} A^{1/2}. \quad (24)$$

The question that can be asked at this point is that are there other symmetric matrix means which correspond to symmetric spaces as midpoint maps on  $P(n, \mathbb{C})$ ? Two other examples are known, these are the arithmetic mean  $(A + B)/2$  and the harmonic mean  $2(A^{-1} + B^{-1})^{-1}$ . The symmetric spaces corresponding to these two means are Euclidean while the symmetric space corresponding to the geometric mean has nonpositive curvature. It has flat and negatively curved de Rham factors.

At this point we begin with the characterization of means that correspond to affine symmetric spaces in general. What we know at this point is that the two functions, which are of each others inverse,  $\log_I(t)$  and  $\exp_I(t)$  exist for all matrix means, as it was proved in Theorem 2.

In [14] and [15] there is an extensive study of affine connections on manifolds. A well known fact is that the affine connection on a manifold can be reconstructed by differentiating the parallel transport:

$$\nabla_{X_p} Y_p = \lim_{t \rightarrow 0} \frac{\Gamma_t^0(\gamma) Y_{\gamma(t)} - Y_{\gamma(0)}}{t},$$

where  $\gamma(t)$  denotes an arbitrary smooth curve emanating from  $p$  in the direction  $X_p = \partial \gamma(t) / \partial t|_{t=0}$  and  $\Gamma_t^s(\gamma) Y$  denotes the parallel transport of the vector field  $Y$  along the curve  $\gamma$  from  $\gamma(t)$  to  $\gamma(s)$ , refer to [14, 15]. The above limit does not depend on the curve itself, only on its initial direction vector and it depends on the vector field  $Y$  in an open neighborhood of  $p$ . On affine symmetric spaces the parallel transport from one point to another along the connecting geodesic is given by the differential of the geodesic symmetries with a negative sign. The geodesic symmetry is given as

$$S_p(q) = \exp_p(-\log_p(q)).$$

On affine symmetric spaces this map is an affine transformation so one can conclude that

$$\Gamma_1^0(\gamma) Y = - \left. \frac{\partial S_{\gamma(1/2)}(\exp_q(Yt))}{\partial t} \right|_{t=0}, \quad (25)$$

where  $\gamma(t)$  is a geodesic connecting  $p = \gamma(0)$  and  $q = \gamma(1)$ .

We have already proved the following formulas for the exponential and logarithm maps at the end of the preceding section

$$\begin{aligned} \exp_p(X) &= p^{1/2} \exp_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} = p \exp_I \left( p^{-1} X \right) \\ \log_p(X) &= p^{1/2} \log_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} = p \log_I \left( p^{-1} X \right). \end{aligned} \quad (26)$$

The above identities already specify the geodesic symmetries with the notation  $S_I(X) = \exp_I(-\log_I(X))$  as

$$S_p(q) = \exp_p(-\log_p(q)) = p^{1/2} S_I(p^{-1/2} q p^{-1/2}) p^{1/2} = p S_I(p^{-1} q). \quad (27)$$

Now we are in position to prove the following

**Theorem 5** *Let  $P(n, \mathbb{C})$  be subset of an affine symmetric space with affine geodesic symmetries given as (27). Then the invariant affine connection has the form*

$$\nabla_{X_p} Y_p = DY[p][X_p] - \frac{\kappa}{2} (X_p p^{-1} Y_p + Y_p p^{-1} X_p), \quad (28)$$

where  $\kappa = S_I''(1)/2$ .

*Proof* We are going to use (25) to obtain the connection (28). We make the assumption that the geodesic symmetries are of the form (27). The functions  $\exp_p(X)$  and  $\log_p(X)$  are of the form (26), where  $\exp_I(t)$  and  $\log_I(t)$  are analytic functions on a disk centered around 0 and 1 respectively. We also have that  $\log_I(1) = 0$ ,  $\exp_I(0) = 1$  and furthermore

$$\left. \frac{\partial \exp_I(t)}{\partial t} \right|_{t=0} = 1.$$

First of all we have to differentiate the map  $S_p(q)$  given in (27) to obtain  $\Gamma_1^0(\gamma)Y = T_{q \rightarrow p}Y$ , where  $\gamma(t)$  is a geodesic connecting  $p = \gamma(0)$  and  $q = \gamma(1)$ .

$$\begin{aligned} \left. \frac{\partial S_p(\exp_q(Yt))}{\partial t} \right|_{t=0} &= \left. \frac{\partial p S_I(p^{-1} \exp_q(Yt))}{\partial t} \right|_{t=0} = \\ &= p D S_I[p^{-1} q] [p^{-1} Y] \end{aligned} \quad (29)$$

We used the fact that  $\partial \exp_q(Yt)/\partial t|_{t=0} = Y$  which is a consequence of  $\exp'_I(0) = 1$ .

Now we are going to differentiate the parallel transport as given by (25) to get back the connection. We use the holomorphic functional calculus to express the Fréchet-differential in (29) as

$$D S_I[X][U] = \frac{1}{2\pi i} \int_g S_I(z) [zI - X]^{-1} U [zI - X]^{-1} dz.$$

It also easy to see that  $D S_I[I][I] = S_I'(1) = -1$ , so we may express the limit (25) by the following differential

$$\nabla_{\gamma'(0)} Y_{\gamma(0)} = - \left. \frac{\partial \gamma(t/2) D S_I[\gamma(t/2)^{-1} \gamma(t)] [\gamma(t/2)^{-1} Y_{\gamma(t)}]}{\partial t} \right|_{t=0} =$$

we massage this further by using the holomorphic functional calculus

$$\begin{aligned}
&= -\frac{\partial}{\partial t} \gamma(t/2) \frac{1}{2\pi i} \int_g S_I(z) [zI - \gamma(t/2)^{-1} \gamma(t)]^{-1} \gamma(t/2)^{-1} Y_{\gamma(t)} \times \\
&\quad [zI - \gamma(t/2)^{-1} \gamma(t)]^{-1} dz \Big|_{t=0} = -\frac{1}{2} \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} D S_I[I][I] - \\
&\quad - \gamma(0) \frac{1}{2\pi i} \int_g S_I(z) \left\{ [zI - I]^{-1} \frac{1}{2} \gamma(0)^{-1} \gamma'(0) [zI - I]^{-1} \gamma(0)^{-1} Y_{\gamma(0)} [zI - I]^{-1} + \right. \\
&\quad + [zI - I]^{-1} \gamma(0)^{-1} Y_{\gamma(0)} [zI - I]^{-1} \frac{1}{2} \gamma(0)^{-1} \gamma'(0) [zI - I]^{-1} + \\
&\quad + [zI - I]^{-1} \left[ -\gamma(0)^{-1} \frac{1}{2} \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} + \gamma(0)^{-1} D Y[\gamma(0)][\gamma'(0)] \right] \times \\
&\quad \left. [zI - I]^{-1} \right\} dz =
\end{aligned}$$

by using the fact that  $D S_I[I][I]$  and  $[zI - I]^{-1}$  commutes with every matrix we get

$$\begin{aligned}
&= -\frac{D S_I[I][I]}{2} \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} - \\
&\quad - \gamma(0) \frac{1}{2\pi i} \int_g \frac{S_I(z) dz}{(z-1)^3} \frac{1}{2} \gamma(0)^{-1} \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} - \\
&\quad - \gamma(0) \frac{1}{2\pi i} \int_g \frac{S_I(z) dz}{(z-1)^3} \frac{1}{2} \gamma(0)^{-1} Y_{\gamma(0)} \gamma'(0) \gamma(0)^{-1} - \\
&\quad - \gamma(0) \frac{1}{2\pi i} \int_g \frac{S_I(z) dz}{(z-1)^2} \left[ -\frac{1}{2} \gamma(0)^{-1} \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} + \gamma(0)^{-1} D Y[\gamma(0)][\gamma'(0)] \right]
\end{aligned}$$

at this point we use the integral representation

$$S_I^{(j)}(1) = \frac{j!}{2\pi i} \int_g \frac{S_I(z)}{(z-1)^{j+1}} dz$$

to further simplify the above.

$$\begin{aligned}
\nabla_{\gamma'(0)} Y_{\gamma(0)} &= -\frac{S_I''(1)}{4} \left[ \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} + Y_{\gamma(0)} \gamma(0)^{-1} \gamma'(0) \right] - \\
&\quad - \frac{S_I'(1)}{2} \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} - \frac{S_I'(1)}{2} \left[ -\gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} + 2 D Y[\gamma(0)][\gamma'(0)] \right] = \\
&= -S_I'(1) D Y[\gamma(0)][\gamma'(0)] - \frac{S_I''(1)}{4} \left[ \gamma'(0) \gamma(0)^{-1} Y_{\gamma(0)} + Y_{\gamma(0)} \gamma(0)^{-1} \gamma'(0) \right].
\end{aligned}$$

So we have that  $\kappa = S_I''(1)/2$ .  $\square$

The above clearly tells us that all symmetric spaces occuring in such a way that their geodesic division maps are matrix means, have invariant affine connections in the form (28). We are going to study these connections as  $\kappa$  being a parameter. We will find out later for which values of  $\kappa$  are these spaces symmetric. Also for arbitrary real  $\kappa$  (28) defines an affine connection with corresponding exponential and logarithm map which are of the form (26) as we will see later. We will also determine if these connections are metric or not.

## 7 Properties of these affine connections

In this section we study the connections

$$\nabla_{X_p} Y_p = DY[p][X_p] - \frac{\kappa}{2} \left( X_p p^{-1} Y_p + Y_p p^{-1} X_p \right) \quad (30)$$

for  $p \in P(n, \mathbb{C})$  and vector fields  $X_p, Y_p \in H(n, \mathbb{C})$  on the smooth manifold  $P(n, \mathbb{C})$  with tangent bundle  $H(n, \mathbb{C})$ . It is easy to see that indeed these connections are affine and analytic for real  $\kappa$ .

We can fix a coordinate frame by taking the basis  $E_i \in H(n, \mathbb{C})$ , where  $i$  indices over the set of distinct hermitian matrices which have zero entries, excluding exactly the entry  $[E_i]_{kl} = 1$  and its transpose  $[E_i]_{lk} = 1$ . If we equip  $H(n, \mathbb{C})$  with the inner product  $\langle X, Y \rangle = \text{Tr}\{XY\}$ , then the  $E_i$  form an orthonormal basis of  $H(n, \mathbb{C})$ . The dimension of  $H(n, \mathbb{C})$  is  $n(n+1)/2$  such as the dimension of the smooth manifold  $P(n, \mathbb{C})$ . In this coordinate frame the Christoffel symbols are given as

$$\Gamma_{ij}^k E_k = -\frac{\kappa}{2} \left( E_i p^{-1} E_j + E_j p^{-1} E_i \right), \quad (31)$$

where we used the Einstein summation convention for repeated covariant and contravariant indices. Given an arbitrary connection  $\nabla$  the geodesic equations corresponding to it are given as

$$\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0 \quad (32)$$

with given initial conditions  $\gamma(0)$  and  $\dot{\gamma}(0)$ , for all  $t \in [0, T)$ . I.e. the curve  $\gamma$  must be parallel along itself.

**Theorem 6** *The geodesic equations corresponding to the affine connections (30) are*

$$\ddot{\gamma} = \kappa \dot{\gamma} \gamma^{-1} \dot{\gamma}. \quad (33)$$

*The solutions of these equations with initial conditions  $\gamma(0) = p$ ,  $\dot{\gamma}(0) = X$  are the following one parameter family of functions*

$$\gamma(t) = \exp_p(Xt) = p^{1/2} \exp_I \left( p^{-1/2} X p^{-1/2} t \right) p^{1/2} \quad (34)$$

where

$$\exp_I(X) = \begin{cases} [(1 - \kappa)X + 1]^{\frac{1}{1-\kappa}} & \text{if } \kappa \neq 1, \\ \exp(X) & \text{else.} \end{cases} \quad (35)$$

*Proof* For the connections (30) it is easy to see that the corresponding  $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$  geodesic equations are (33).

Let us first consider the case when  $\gamma(0) = p = I = \dot{\gamma}(0) = X$ . Then it is enough to solve the equation (33) for real numbers. Therefore the equation takes the form

$$\exp_I''(t) = \kappa \exp_I'(t)^2 \exp_I(t)^{-1}. \quad (36)$$

If we transform the equation to the inverse function of  $\exp_I(t)$  which will be the logarithm map  $\log_I(t)$ , then we get a separable first order differential equation of the form

$$\log_I''(t) = -\kappa \log_I'(t) t^{-1}.$$

Solving the above we get the logarithm map as

$$\log_I(X) = \begin{cases} \frac{X^{1-\kappa}-1}{1-\kappa} & \text{if } \kappa \neq 1, \\ \log(X) & \text{else.} \end{cases}$$

From this by inverting the above function we get the assertion for real numbers.

Now we check by substitution into (33) that the curve

$$\gamma(t) = p^{1/2} \exp_I \left( p^{-1/2} X p^{-1/2} t \right) p^{1/2}$$

is also a solution of the equations (33), since the function  $\exp_I$  is analytic. Indeed

$$\dot{\gamma}(t) = X p^{-1/2} \exp'_I \left( p^{-1/2} X p^{-1/2} t \right) p^{1/2} = p^{1/2} \exp'_I \left( p^{-1/2} X p^{-1/2} t \right) p^{-1/2} X$$

$$\ddot{\gamma}(t) = X p^{-1/2} \exp''_I \left( p^{-1/2} X p^{-1/2} t \right) p^{-1/2} X$$

and after substitution we get

$$\begin{aligned} X p^{-1/2} \exp''_I \left( p^{-1/2} X p^{-1/2} t \right) p^{-1/2} X &= \kappa X p^{-1/2} \exp'_I \left( p^{-1/2} X p^{-1/2} t \right) \times \\ &\quad \exp_I \left( p^{-1/2} X p^{-1/2} t \right)^{-1} \exp'_I \left( p^{-1/2} X p^{-1/2} t \right) p^{-1/2} X \end{aligned}$$

which is fulfilled since

$$\exp''_I \left( p^{-1/2} X p^{-1/2} t \right) = \kappa \exp'_I \left( p^{-1/2} X p^{-1/2} t \right)^2 \exp_I \left( p^{-1/2} X p^{-1/2} t \right)^{-1}$$

holds by the functional calculus for  $\exp_I$  and its derivatives and (36).  $\square$

**Corollary 1** *The exponential and logarithm map for the affine connections (30) are given in the form*

$$\begin{aligned} \exp_p(X) &= p^{1/2} \exp_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} \\ \log_p(X) &= p^{1/2} \log_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2}, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \exp_I(X) &= \begin{cases} [(1-\kappa)X + 1]^{\frac{1}{1-\kappa}} & \text{if } \kappa \neq 1, \\ \exp(X) & \text{else,} \end{cases} \\ \log_I(X) &= \begin{cases} \frac{X^{1-\kappa}-1}{1-\kappa} & \text{if } \kappa \neq 1, \\ \log(X) & \text{else.} \end{cases} \end{aligned} \tag{38}$$

The affine matrix means which induce these affinely connected manifolds are

$$\begin{aligned} M_t(X, Y) &= \exp_X(t \log_X(Y)) = \\ &= \begin{cases} X^{1/2} \left[ (1-t)I + t \left( X^{-1/2} Y X^{-1/2} \right)^{1-\kappa} \right]^{\frac{1}{1-\kappa}} X^{1/2} & \text{if } \kappa \neq 1, \\ X^{1/2} \left( X^{-1/2} Y X^{-1/2} \right)^t X^{1/2} & \text{else,} \end{cases} \end{aligned} \tag{39}$$

if  $\kappa \in [0, 2]$ , for other values of  $\kappa$  the functions (39) are not matrix means.

*Proof* The first part of the assertion is clear, the second part follows from the fact that (39) are matrix means if and only if  $\kappa \in [0, 2]$ , refer to Example 1.  $\square$

*Remark 2* The one-parameter family of affine matrix means (39) seems to have a singularity at  $\kappa = 1$ , however it is known that the singularity is removable and indeed as  $\kappa \rightarrow 1$  we get the matrix geometric mean as the limit. This phenomenon has already been investigated in [26]. In that paper the same one-parameter family of matrix means were considered under the name of matrix power means.

If  $\kappa = 0$  we get back the arithmetic mean as the midpoint operation, and the weighted arithmetic mean

$$A_t(A, B) = (1 - t)A + tB \quad (40)$$

is the geodesic line connecting  $A$  and  $B$  with respect to the metric  $\langle X, Y \rangle_p = \text{Tr}\{X^*Y\}$ . If  $\kappa = 2$  we get back the harmonic mean as the midpoint operation, and the weighted harmonic mean

$$H_t(A, B) = \left( (1 - t)A^{-1} + tB^{-1} \right)^{-1} \quad (41)$$

is also a geodesic with respect to the metric  $\langle X, Y \rangle_p = \text{Tr}\{p^{-2}Xp^{-2}Y\}$ . We have already mentioned that the second metric is isometric to the first one, so it is also Euclidean.

In the case when  $\kappa = 1$  the midpoint is the geometric mean and the geodesics are given by the weighted geometric mean

$$G_t(A, B) = A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^t A^{1/2}. \quad (42)$$

The corresponding Riemannian metric is  $\langle X, Y \rangle_p = \text{Tr}\{p^{-1}Xp^{-1}Y\}$ . This manifold, which is the symmetric space  $GL(n, \mathbb{C})/U(n, \mathbb{C})$ , is nonpositively curved while the other two has zero curvature.

In the paper [26] matrix power means  $P_s(w_1, \dots, w_k; A_1, \dots, A_k)$  are defined as the unique positive definite solution of the equations

$$X = \sum_{i=1}^k w_i G_s(X, A_i) \quad (43)$$

where  $s \in [-1, 1]$ ,  $w_i > 0$ ,  $\sum_{i=1}^k w_i = 1$  and  $A_i \in P(n, \mathbb{C})$ . Existence and uniqueness of the solutions follow from the fact that the function

$$f(X) = \sum_{i=1}^k w_i G_s(X, A_i)$$

is a strict contraction for  $s \in [-1, 1]$ ,  $s \neq 0$  with respect to Thompson's part metric [26]. In the case  $k = 2$  we get back the affine matrix means (39) with  $s = \kappa - 1$  and  $t = w_2$ .

**Corollary 2** *With the identification  $s = \kappa - 1$ , the two-variable matrix power means  $P_s(w_1, w_2; A_1, A_2)$  are geodesic lines, with  $w_2$  being the arc-length parameter, of the affinely connected spaces with affine connections (30).*

The arithmetic (40), harmonic (41) and geometric (42) means have nice characterizations and extensions to several variables as the center of mass or Karcher mean of the corresponding manifolds [31, 32, 7, 26]. I.e.

$$A(w_1, \dots, w_k; A_1, \dots, A_k) = \arg \min_{X \in P(n, \mathbb{C})} \sum_{i=1}^n w_i d^2(X, A_i), \quad (44)$$

where  $d(\cdot, \cdot)$  is a Riemannian metric given as

$$d^2(X, Y) = \langle \log_X(Y), \log_X(Y) \rangle_X \quad (45)$$

where  $\langle \cdot, \cdot \rangle_X$  is one of the corresponding metrics given above for the arithmetic (40), harmonic (41) and geometric (42) means, and  $\log_X(Y)$  are the corresponding logarithm maps (37) (for  $\kappa = 0, 2, 1$  respectively). It is well known that in geodesically convex neighborhoods on a Riemannian manifold (44) has a unique solution [21, 31]. The solution can be expressed by taking the gradient of the cost function on the right hand side of (44) [21] and then one arrives at

$$\sum_{i=1}^n w_i \log_X(A_i) = 0. \quad (46)$$

The unique solution of this equation can be expressed in closed form in the case of the arithmetic and harmonic means, since the corresponding manifolds are Euclidean. The solutions are just the multivariable weighted arithmetic  $\sum_{i=1}^k w_i A_i$  and harmonic means  $\left(\sum_{i=1}^k w_i A_i^{-1}\right)^{-1}$  [32]. These functions are monotone in their variables with respect to the positive definite order and have some other desirable properties [26]. These two cases are well known and of less interest, however the same situation is of much more interest in the case of the geometric mean. In this case the unique solution of the minimization problem (44) cannot be expressed easily in closed form since the corresponding Riemannian manifold is no longer flat. The corresponding equation for the gradient (46) is given in the form

$$\sum_{i=1}^n w_i \log(X^{1/2} A_i^{-1} X^{1/2}) = 0 \quad (47)$$

and usually this equation is called the Karcher equation [26] and the corresponding unique positive definite solution (44) the Karcher mean. Several properties of this mean were open problems, for example its monotonicity with respect to the positive definite order, however this and other key properties of the mean were proved by using different techniques [25, 26, 8]. One of the techniques given in [26] is based on the matrix power means  $P_s(w_1, \dots, w_k; A_1, \dots, A_k)$ . These means are given as the unique positive definite solutions of (43). The following result provides a geometric characterization of matrix power means.

**Proposition 9** *The matrix power means  $P_s(w_1, \dots, w_k; A_1, \dots, A_k)$  for  $s \in [-1, 1]$  are the unique positive definite solutions of the Karcher equations*

$$\sum_{i=1}^n w_i \log_X(A_i) = 0 \quad (48)$$

where  $\log_X(A_i)$  are the logarithm maps (37) corresponding to the affine family (30) with parameter identification  $s = \kappa - 1$ .

*Proof* The defining equation (43) of matrix power means with  $s \neq 0$  is equivalent to

$$\begin{aligned} \sum_{i=1}^k w_i (G_s(X, A_i) - X) &= 0 \\ \sum_{i=1}^n w_i X^{1/2} \left[ \left( X^{-1/2} A_i X^{-1/2} \right)^s - I \right] X^{1/2} &= 0 \\ \sum_{i=1}^n w_i X^{1/2} \frac{\left( X^{-1/2} A_i X^{-1/2} \right)^s - I}{s} X^{1/2} &= 0, \end{aligned}$$

which is by (37) equivalent to

$$\sum_{i=1}^n w_i \log_X(A_i) = 0.$$

The case  $s = 0$  is just the case (47).  $\square$

*Remark 3* By the continuity of fixed points of pointwise continuous families of strict contractions [26], the unique solution of (48) varies continuously with respect to the parameter  $s$ . The singularity at  $s = 0$  is known to be removable and the limit is just the Karcher mean [26].

Now since the Karcher equations (48) admit unique positive definite solutions the obvious question arises whether there are Riemannian metrics corresponding to other values of  $\kappa$ ? Also for what other values of  $\kappa$  is the manifold  $P(n, \mathbb{C})$  with affine connection (30) a symmetric space? The full solution of these questions requires the study of the curvature tensors and holonomy groups which is postponed to the last section. At this point we prove some other results which gets us closer to this metrization problem of the affine connections (30). First of all we compute the parallel transport over a geodesic connecting an arbitrary point and the identity. The parallel transport of a vector  $Y_{\gamma(0)}$  given in the tangent space at  $\gamma(0)$  with respect to the connection  $\nabla$  along the curve  $\gamma(t)$  is defined to be the vector field  $Y_{\gamma(t)}$  which is the solution of the ODE

$$\nabla_{\dot{\gamma}(t)} Y = 0.$$

**Proposition 10** *Let  $c(t)$  be a geodesic with respect to the connection (30) and  $c(0) = I, c(1) = p$ . Then the unique solution of  $\nabla_{\dot{c}(t)} Y = 0$  with respect to the connection (30) and the initial condition  $Y_{c(0)} = Y_0$  is the vector field*

$$Y(t) = c(t)^{\frac{\kappa}{2}} Y_0 c(t)^{\frac{\kappa}{2}}. \quad (49)$$

*Proof* We have to integrate the equation  $\nabla_{c'(t)} Y_{c(t)} = 0$ . This is equivalent to

$$DY[c(t)][c'(t)] - \frac{\kappa}{2} \left( c'(t) c(t)^{-1} Y_{c(t)} + Y_{c(t)} c(t)^{-1} c'(t) \right) = 0.$$

We are looking for the solution  $Y_{c(t)} = Y(t)$  in the form

$$Y(t) = f(c(t)) Y_0 f(c(t)),$$



for some analytic function  $f(x)$ . We have for the Fréchet-differential

$$DY[c(t)][c'(t)] = \frac{dY(t)}{dt} = \frac{df(c(t))}{dt} Y_0 f(c(t)) + f(c(t)) Y_0 \frac{df(c(t))}{dt}.$$

Now substituting into the equation of the parallel transport above, we get

$$\frac{df(c(t))}{dt} Y_0 f(c(t)) + f(c(t)) Y_0 \frac{df(c(t))}{dt} = \frac{\kappa}{2} \left( c'(t) c(t)^{-1} Y_{c(t)} + Y_{c(t)} c(t)^{-1} c'(t) \right).$$

Since  $c(t) = \exp_I(t \log_I(p))$ , it has a power series expansion, as has  $f(x)$ , so we have by commutativity that

$$\frac{\kappa}{2} c'(t) c(t)^{-1} f(c(t)) = \frac{df(c(t))}{dt} = Df[c(t)][c'(t)] = f'(c) c'(t).$$

Since everything on the left and right hand side commutes with one another, we arrive at the following separable differential equation

$$\frac{\kappa}{2} c^{-1} = f'(c) f(c)^{-1},$$

which has its solution in the form  $f(c) = c^{\kappa/2}$ .  $\square$

On a Riemannian manifold the length of vectors are left invariant by parallel transports with respect to the Levi-Civita connection due to the Fundamental Theorem of Riemannian geometry [22]. It is easy to check that the connections (30) are symmetric and torsion free so any of them can possibly be a Levi-Civita connection of a Riemannian manifold. So by the above proposition we should look for the Riemannian metrics in the form

$$\left\langle p^{-\kappa/2} X p^{-\kappa/2}, p^{-\kappa/2} Y p^{-\kappa/2} \right\rangle_{\kappa} \quad (50)$$

for some positive definite bilinear forms  $\langle \cdot, \cdot \rangle_{\kappa}$  given on the tangent space at  $I$ . In the next section we prove that all affine matrix means are actually matrix power means, i.e. we do not have to look for other connections than (30).

## 8 The classification of affine matrix means

Due to Proposition 8 we have the exponential and logarithm map of affine matrix means in the form

$$\begin{aligned} \exp_p(X) &= p^{1/2} \exp_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} \\ \log_p(X) &= p^{1/2} \log_I \left( p^{-1/2} X p^{-1/2} \right) p^{1/2} \end{aligned} \quad (51)$$

for  $p \in P(n, \mathbb{C})$ , where  $\exp_I(X)$  and  $\log_I(X)$  are analytic functions. The function  $\exp_I : H(n, \mathbb{C}) \mapsto P(n, \mathbb{C})$  and  $\log_I(X)$  is its inverse,  $\log'_I(I) = I, \exp'_I(0) = I, \log_I(I) = 0, \exp_I(0) = I$ . Suppose that (51) represent the exponential and logarithm map of an affinely connected manifold. Then the analytic function  $\exp_I(t)$  is the solution of some geodesic equations

$$\exp''_I(t) + \Gamma(\exp'_I(t), \exp'_I(t), \exp_I(t)) = 0,$$

where  $\Gamma(\cdot, \cdot, \cdot) : H(n, \mathbb{C}) \times H(n, \mathbb{C}) \times P(n, \mathbb{C}) \mapsto H(n, \mathbb{C})$  is a smooth function in all variables and linear in the first two, representing the Christoffel symbols of an affine connection. By Proposition 15 and Corollary 16 of Chapter 6 in [33] we know that connections which have the same torsion and geodesics are identical and for an arbitrary connection there is a unique connection with vanishing torsion and with the same geodesics. If we have an affine connection with non-symmetric Christoffel symbols  $\Gamma_{jk}^i$ , it has the same geodesics as its symmetric part  $\frac{\Gamma_{jk}^i + \Gamma_{kj}^i}{2}$ , so without loss of generality we can assume in our case that all connections are symmetric, so we will be considering mappings  $\Gamma(\cdot, \cdot, \cdot)$  which are symmetric in their first two arguments.

**Proposition 11** *Suppose that  $\Gamma(\cdot, \cdot, \cdot), \exp_I(\cdot), \exp_p(\cdot)$  are functions given with the above properties. Then*

$$\Gamma(X, X, p) = p^{1/2} \Gamma\left(p^{-1/2} X p^{-1/2}, p^{-1/2} X p^{-1/2}, I\right) p^{1/2} \quad (52)$$

for  $p \in P(n, \mathbb{C})$  and  $X \in H(n, \mathbb{C})$ .

*Proof* Let  $\gamma(t) = \exp_I\left(p^{-1/2} X p^{-1/2} t\right)$ . Since  $\exp_I$  is an analytic function we have

$$\begin{aligned} \dot{\gamma}(t) &= p^{-1/2} X p^{-1/2} \exp_I'\left(p^{-1/2} X p^{-1/2} t\right) \\ \ddot{\gamma}(t) &= p^{-1/2} X p^{-1/2} \exp_I''\left(p^{-1/2} X p^{-1/2} t\right) p^{-1/2} X p^{-1/2} \end{aligned}$$

and other formulas hold for  $\dot{\gamma}(t)$  and  $\ddot{\gamma}(t)$  similarly to the second part of the proof of Theorem 6. By the geodesic equations we have

$$\begin{aligned} \ddot{\gamma}(t) &= -\Gamma(\dot{\gamma}(t), \dot{\gamma}(t), \gamma(t)) \\ X p^{-1/2} \exp_I''\left(p^{-1/2} X p^{-1/2} t\right) p^{-1/2} X &= -p^{1/2} \Gamma\left(p^{-1/2} X p^{-1/2} \exp_I'\left(p^{-1/2} X p^{-1/2} t\right), p^{-1/2} X p^{-1/2} \exp_I'\left(p^{-1/2} X p^{-1/2} t\right), \exp_I\left(p^{-1/2} X p^{-1/2} t\right)\right) p^{1/2}. \end{aligned}$$

If we consider the geodesic equations for  $\gamma(t) = \exp_p(Xt)$  we get

$$\begin{aligned} X p^{-1/2} \exp_I''\left(p^{-1/2} X p^{-1/2} t\right) p^{-1/2} X &= -\Gamma\left(X p^{-1/2} \exp_I'\left(p^{-1/2} X p^{-1/2} t\right) p^{1/2}, p^{1/2} \exp_I'\left(p^{-1/2} X p^{-1/2} t\right) p^{-1/2} X, p^{1/2} \exp_I\left(p^{-1/2} X p^{-1/2} t\right) p^{1/2}\right). \end{aligned}$$

The left hand sides of the two equations above are the same so as the right hand sides. Taking  $t = 0$  and that  $\exp_I'(0) = I, \exp_I(0) = I$  we get for all  $p \in P(n, \mathbb{C}), X \in H(n, \mathbb{C})$  that

$$\begin{aligned} p^{1/2} \Gamma\left(p^{-1/2} X p^{-1/2}, p^{-1/2} X p^{-1/2}, I\right) p^{1/2} &= \\ &= \Gamma(X, X, p), \end{aligned}$$

which proves the assertion.  $\square$

By the above result we have just reduced the problem of characterizing  $\Gamma(X, X, p)$  to the characterization of  $\Gamma(X, X, I)$ . Now we will show that  $\Gamma(X, X, p)$  is invariant under similarity transformations.

**Proposition 12** For all  $p \in P(n, \mathbb{C})$  and  $X \in H(n, \mathbb{C})$  and invertible  $S$  we have

$$\Gamma(SXS^{-1}, SXS^{-1}, SpS^{-1}) = S\Gamma(X, X, p)S^{-1}. \quad (53)$$

*Proof* We have by the geodesic equations

$$\begin{aligned} X^2 \exp_I''(Xt) &= -\Gamma(X \exp_I'(Xt), X \exp_I'(Xt), \exp_I(Xt)) \\ SX^2 \exp_I''(Xt)S^{-1} &= -S\Gamma(X \exp_I'(Xt), X \exp_I'(Xt), \exp_I(Xt))S^{-1}. \end{aligned}$$

Similarly if we consider the geodesic equations for the curve  $\gamma(t) = \exp_I(SXS^{-1}t)$  we get

$$\begin{aligned} SX^2S^{-1} \exp_I''(SX S^{-1}t) &= -\Gamma(SXS^{-1} \exp_I'(SX S^{-1}t), SX S^{-1} \exp_I'(SX S^{-1}t), \\ &\quad \exp_I(SX S^{-1}t)) \\ SX^2 \exp_I''(Xt)S^{-1} &= -\Gamma(SX \exp_I'(Xt)S^{-1}, SX \exp_I'(Xt)S^{-1}, \\ &\quad S \exp_I(Xt)S^{-1}). \end{aligned}$$

Again since the above two equations are identical we get the assertion.  $\square$

By the above proposition we have for hermitian  $X$  that

$$\Gamma(X, X, I) = U\Gamma(D, D, I)U^*, \quad (54)$$

for some diagonal  $D$  and unitary  $U$ , so it is enough to characterize  $\Gamma(X, X, I)$  for diagonal  $X$ .

**Theorem 7** Let  $D$  be diagonal with real coefficients. Then

$$\Gamma(D, D, I) = -cD^2, \quad (55)$$

for some real valued constant  $c$ .

*Proof* First we will show that  $\Gamma(I, I, I) = cI$  for some real constant  $c$ . Consider the case when  $\gamma(t) = \exp_I(\lambda It)$  for some real  $\lambda$ . Then by the geodesic equations for  $\gamma(t)$  we have

$$\lambda^2 \exp_I''(\lambda It) = -\Gamma(\lambda \exp_I'(\lambda It), \lambda \exp_I'(\lambda It), \exp_I(\lambda It)).$$

By linearity of  $\Gamma(\cdot, \cdot, \cdot)$  in the first two variables, this is equivalent to

$$\lambda^2 \exp_I''(\lambda It) = -\lambda^2 \Gamma(\exp_I'(\lambda It), \exp_I'(\lambda It), \exp_I(\lambda It)).$$

Letting  $t = 0$  we get

$$cI = -\Gamma(I, I, I),$$

where  $c = \exp_I''(0)$  is a real number, since  $\exp_I : H(n, \mathbb{C}) \mapsto P(n, \mathbb{C})$  is an analytic function with real coefficients in its Taylor series.

The next step is to show that for a projection  $P = P^2 = P^*$  we have  $\Gamma(P, P, I) = -cP$ . Consider again  $\gamma(t) = \exp_I(Pt)$ . Then the geodesic equations read

$$P^2 \exp_I''(Pt) = -\Gamma(P \exp_I'(Pt), P \exp_I'(Pt), \exp_I(Pt)).$$

Since  $P^2 = P$  and again letting  $t = 0$  we get

$$cP = -\Gamma(P, P, I),$$

where  $c$  is trivially the same constant as determined above for  $\Gamma(I, I, I)$ . Now suppose that we have two mutually orthogonal projections  $P_1, P_2$  such that  $P_1 P_2 = 0$ . Then we have for the projection  $P_1 + P_2$  using linearity of  $\Gamma(\cdot, \cdot, \cdot)$  in the first two variables that

$$\begin{aligned} \Gamma(P_1, P_1, I) + \Gamma(P_2, P_2, I) &= -c(P_1 + P_2) = \Gamma(P_1 + P_2, P_1 + P_2, I) = \\ &= \Gamma(P_1, P_1, I) + \Gamma(P_1, P_2, I) + \Gamma(P_2, P_1, I) + \Gamma(P_2, P_2, I), \end{aligned}$$

which yields that for mutually orthogonal projections  $P_1, P_2$  we get the orthogonality relation

$$\Gamma(P_1, P_2, I) = 0.$$

Finally since a diagonal  $D$  can be written as  $D = \sum_i \lambda_i P_i$  for mutually orthogonal projections  $P_i$ , we have

$$\begin{aligned} \Gamma(D, D, I) &= \Gamma\left(\sum_i \lambda_i P_i, \sum_i \lambda_i P_i, I\right) = \\ &= \sum_i \lambda_i^2 \Gamma(P_i, P_i, I) = -\sum_i \lambda_i^2 c P_i = \\ &= -c D^2, \end{aligned}$$

which is what needed to be shown.  $\square$

The above three theorems with the other preceeding results presented here, lead us to the concluding

**Theorem 8** *All affine matrix means  $M_t(X, Y)$  are of the form*

$$M(X, Y) = \begin{cases} X^{1/2} \left[ (1-t)I + t \left( X^{-1/2} Y X^{-1/2} \right)^{1-\kappa} \right]^{\frac{1}{1-\kappa}} X^{1/2} & \text{if } \kappa \neq 1, \\ X^{1/2} \left( X^{-1/2} Y X^{-1/2} \right)^t X^{1/2} & \text{if } \kappa = 1, \end{cases} \quad (56)$$

where  $0 \leq \kappa \leq 2$ . The symmetric affine connections corresponding to these means are

$$\nabla_{X_p} Y_p = DY[p][X_p] - \frac{\kappa}{2} \left( X_p p^{-1} Y_p + Y_p p^{-1} X_p \right). \quad (57)$$

*Proof* By Proposition 11, 12 and Theorem 7 we have that the functions  $\Gamma(\cdot, \cdot, \cdot) : H(n, \mathbb{C}) \times H(n, \mathbb{C}) \times P(n, \mathbb{C}) \mapsto H(n, \mathbb{C})$  representing the Christoffel symbols are of the form

$$\Gamma(X, X, p) = -c X p^{-1} X. \quad (58)$$

This formula determines the functions that are the symmetric parts of the possible connections, and these connections have geodesics determined by Theorem 6 in the form (56). Again by Proposition 15 and Corollary 16 of Chapter 6 in [33] we know that connections which have the same torsion and geodesics are identical and for an arbitrary connection there is a unique connection with vanishing torsion and with the same geodesics. So in other words since the connections (57) are

symmetric, affine and have the same geodesics, therefore they give *the* sought symmetric connections for each  $\kappa$  if we choose  $c = \kappa$ .

The corresponding geodesics are given in (39), and these are matrix means if and only if  $\kappa \in [0, 2]$ , since the representing functions  $f(t)$  in (4) turn out to be operator monotone only in these cases due to Example 1.  $\square$

The above result gives us the complete classification of affine matrix means. So now we can concentrate only on the connections (57). In the next section we solve the metrization problem of these connections.

## 9 The holonomy groups and metrizability of the affine family

Let  $W$  be a smooth connected manifold with an affine connection  $\nabla$ . The holonomy group  $\mathcal{H}_p(\nabla)$  of the connection  $\nabla$  at point  $p \in W$  is defined to be the set of all linear automorphisms of the tangent space  $T_p W$  at  $p$  induced by parallel transports along  $p$  based closed rectifiable curves. If  $W$  is simply connected then  $\mathcal{H}_p(\nabla)$  is known to be a Lie subgroup of  $\text{End}(T_p W)$  [27]. In case of non-simply connectedness the restricted holonomy group  $\hat{\mathcal{H}}_p(\nabla)$  is defined as the normal subgroup of  $\mathcal{H}_p(\nabla)$  which is induced by closed rectifiable curves homotopic to zero, see Chapter II Section 4 in [22] for more detailed information. Let  $\mathfrak{h}_p(\nabla)$  and  $\hat{\mathfrak{h}}_p(\nabla)$  denote the Lie algebra of  $\mathcal{H}_p(\nabla)$  and  $\hat{\mathcal{H}}_p(\nabla)$  respectively. The holonomy group  $\mathcal{H}_p(\nabla)$  is known to be an invariant of the connected manifold  $W$ , since  $\mathcal{H}_p(\nabla)$  is conjugate to every other  $\mathcal{H}_q(\nabla)$  by parallel transports.

Now suppose that the connection  $\nabla$  is real analytic. Then by Theorem 10.8 of Chapter II and Theorem 9.2 of Chapter III in [22],  $\hat{\mathfrak{h}}_p(\nabla)$  is generated by the successive covariant differentials  $\nabla^r R$ ,  $r = 0, 1, 2, \dots$  at the point  $p$  where  $R(X, Y)$  denotes the curvature endomorphism of the connection  $\nabla$ . This is a version of Ambrose-Singer's theorem of Kobayashi-Nomizu. The curvature tensor  $R$  is defined as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

or expressed in local coordinate system with the Christoffel symbols  $\Gamma_{jk}^i$  as

$$R_{jkl}^i = \frac{\partial \Gamma_{lj}^i}{\partial x^k} - \frac{\partial \Gamma_{kj}^i}{\partial x^l} + \Gamma_{km}^i \Gamma_{lj}^m - \Gamma_{lm}^i \Gamma_{kj}^m. \quad (59)$$

Suppose now that the connection  $\nabla$  is torsion-free, i.e.

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0 \quad (60)$$

for all vector fields  $X, Y$  or equivalently  $\Gamma_{jk}^i = \Gamma_{kj}^i$  everywhere. Then  $\nabla$  is the Levi-Civita connection of a Riemannian metric if and only if the corresponding holonomy group  $\hat{\mathcal{H}}_p(\nabla)$  is a compact Lie group. More generally there exists a non-degenerate  $\nabla$  invariant bilinear form  $\langle \cdot, \cdot \rangle_p$  if and only if  $\hat{\mathcal{H}}_p(\nabla)$  leaves  $\langle \cdot, \cdot \rangle_p$  invariant.

In [27] all possible irreducible holonomy groups of torsion-free affine connections are classified, so in principle we know what kind of groups can occur, at least in the reducible case. Again we are interested in the connections

$$\nabla_{X_p} Y_p = DY[p][X_p] - \frac{\kappa}{2} \left( X_p p^{-1} Y_p + Y_p p^{-1} X_p \right). \quad (61)$$

These connections are real analytic, torsion-free and the corresponding manifold  $P(n, \mathbb{C})$  is analytic simply connected. So to answer the question of metrizable we have to determine the holonomy groups  $\mathcal{H}_p(\nabla)$ .

In our case it turns out that

$$\begin{aligned} \Gamma_{jk}^i E_i &= -\frac{\kappa}{2}(E_j p^{-1} E_k + E_k p^{-1} E_j) \\ R_{jkl}^i E_i &= \left(\frac{\kappa}{2} - \frac{\kappa^2}{4}\right) p \left[ p^{-1} E_j, \left[ p^{-1} E_k, p^{-1} E_l \right] \right], \end{aligned} \quad (62)$$

where the  $E_i$  form the standard basis of the vector space of  $H(n, \mathbb{C})$  and  $[\cdot, \cdot]$  is the commutator. Note that the tangent space is  $H(n, \mathbb{C})$ , so the left hand sides are in  $H(n, \mathbb{C})$ . In order to determine which of these manifolds are symmetric spaces it is sufficient to calculate the covariant differential  $R_{jkl;m}^s$ , since it vanishes everywhere if and only if the underlying manifold is a symmetric space [15]. Given the basis  $E_i$  for  $H(n, \mathbb{C})$  we have the identities

$$\begin{aligned} R(X, Y; A_1, \dots, A_r)Z &= \sum_{i,j,k,l_1,\dots,l_r,m} R_{jkm;l_1,\dots,l_r}^i E_i X^j Y^k Z^m A_1^{l_1} \dots A_r^{l_r} \\ R_{jkm;l_1,\dots,l_{r+1}}^i &= \frac{\partial}{\partial x^{l_{r+1}}} R_{jkm;l_1,\dots,l_r}^i + \Gamma_{sl_{r+1}}^i R_{jkm;l_1,\dots,l_r}^s \\ &\quad - \sum_{\mu} \Gamma_{l_{r+1}\mu}^s R_{jkm;l_1,\dots,s,\dots,l_r}^i \end{aligned}$$

where indices after ; denote covariant differentiation. Now we prove an analogue of Lemma 1 given in the proof of Theorem 9.2 of Chapter III [22].

**Theorem 9** *Let the smooth connected manifold  $P(n, \mathbb{C})$  be equipped with real analytic connection  $\nabla$  and curvature tensor given by (62) with  $\kappa \in \mathbb{R}$ . Then*

$$R(X, Y; A_1, \dots, A_r)Z = (1 - \kappa)^r D(R(X, Y)Z)[p][A_1, \dots, A_r] \quad (63)$$

where  $D(R(X, Y)Z)[p][A_1, \dots, A_r]$  denotes the  $r$ -th Fréchet differential of the map  $R(X, Y)Z$  at the point  $p \in P(n, \mathbb{C})$  in the directions  $A_i \in H(n, \mathbb{C})$ .

*Proof* The proof is based on writing  $R(X, Y)Z$  and its subsequent covariant differentials in essentially two equivalent ways. First of all note that

$$\frac{\partial}{\partial x^i} p^{-1} = D(x^{-1})[p][E_i] = -p^{-1} E_i p^{-1}, \quad (64)$$

so the differential operator  $\frac{\partial}{\partial x^i}$  is equivalent to Fréchet differentiation at  $p$  in the direction of  $E_i$ , also

$$\begin{aligned} R(X, Y)Z &= \left(\frac{\kappa}{2} - \frac{\kappa^2}{4}\right) p \left[ p^{-1} Z, \left[ p^{-1} X, p^{-1} Y \right] \right] \\ &= \left(\frac{\kappa}{2} - \frac{\kappa^2}{4}\right) \left\{ Z \left[ p^{-1} X, p^{-1} Y \right] + \left[ Y p^{-1}, X p^{-1} \right] Z \right\} \\ &= \left(\frac{\kappa}{2} - \frac{\kappa^2}{4}\right) \left[ Z p^{-1}, \left[ X p^{-1}, Y p^{-1} \right] \right] p. \end{aligned} \quad (65)$$

Using index-less notation and the linearity of  $R(X, Y; A_1, \dots, A_r)Z$  we have

$$\begin{aligned}
R(X, Y; A_1, \dots, A_{r+1})Z &= \nabla_{A_{r+1}}(R(X, Y; A_1, \dots, A_r)Z) \\
&- \frac{\kappa}{2} \left\{ A_{r+1}p^{-1}R(X, Y; A_1, \dots, A_r)Z + R(X, Y; A_1, \dots, A_r)Zp^{-1}A_{r+1} \right. \\
&- R(A_{r+1}p^{-1}X + Xp^{-1}A_{r+1}, Y; A_1, \dots, A_r)Z \\
&- R(X, A_{r+1}p^{-1}Y + Yp^{-1}A_{r+1}; A_1, \dots, A_r)Z \\
&- R(X, Y; A_1, \dots, A_r)(A_{r+1}p^{-1}Z + Zp^{-1}A_{r+1}) \\
&\left. - \sum_{i=1}^r R(X, Y; A_1, \dots, A_{r+1}p^{-1}A_i + A_ip^{-1}A_{r+1}, \dots, A_r)Z \right\}. \tag{66}
\end{aligned}$$

Again the first term in the above equation is equivalent to

$$(A_{r+1})^s \frac{\partial}{\partial x^s} R(X, Y; A_1, \dots, A_r)Z = D(R(X, Y; A_1, \dots, A_r)Z)[p][A_{r+1}]. \tag{67}$$

*Claim*  $R(X, Y; A_1, \dots, A_r)Z$  is the linear combination of terms  $pS$ , where  $S$  is some word which is a product of the terms  $p^{-1}X, p^{-1}Y, p^{-1}A_1, \dots, p^{-1}A_r$  of the first order.

*Proof (of the claim)* We prove by induction. For  $r = 0$  it clearly holds by the first equality in (65). Suppose that it holds for some  $r$ . Then by (66) it is easy to see that it holds for  $r + 1$ , due to (64), the linearity of  $R(X, Y; A_1, \dots, A_r)Z$  and the product rule of Fréchet differentiation.

By the claim  $R(X, Y; A_1, \dots, A_r)Z$  is the linear combination of terms  $pS$ , therefore by linearity, (67) and (64) we have

$$\begin{aligned}
\nabla_{A_{r+1}}(R(X, Y; A_1, \dots, A_r)Z) &= A_{r+1}p^{-1}R(X, Y; A_1, \dots, A_r)Z \\
&- R(A_{r+1}p^{-1}X, Y; A_1, \dots, A_r)Z - R(X, A_{r+1}p^{-1}Y; A_1, \dots, A_r)Z \\
&- R(X, Y; A_1, \dots, A_r)(A_{r+1}p^{-1}Z) - \sum_{i=1}^r R(X, Y; A_1, \dots, A_{r+1}p^{-1}A_i, \dots, A_r)Z. \tag{68}
\end{aligned}$$

Combining the above we arrive at a version of (66):

$$\begin{aligned}
R(X, Y; A_1, \dots, A_{r+1})Z &= \\
&= \left(1 - \frac{\kappa}{2}\right) \left\{ A_{r+1}p^{-1}R(X, Y; A_1, \dots, A_r)Z - R(A_{r+1}p^{-1}X, Y; A_1, \dots, A_r)Z \right. \\
&- R(X, A_{r+1}p^{-1}Y; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)(A_{r+1}p^{-1}Z) \\
&- \sum_{i=1}^r R(X, Y; A_1, \dots, A_{r+1}p^{-1}A_i, \dots, A_r)Z \left. \right\} \\
&- \frac{\kappa}{2} \left\{ R(X, Y; A_1, \dots, A_r)Zp^{-1}A_{r+1} - R(Xp^{-1}A_{r+1}, Y; A_1, \dots, A_r)Z \right. \\
&- R(X, Yp^{-1}A_{r+1}; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)(Zp^{-1}A_{r+1}) \\
&\left. - \sum_{i=1}^r R(X, Y; A_1, \dots, A_ip^{-1}A_{r+1}, \dots, A_r)Z \right\},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
R(X, Y; A_1, \dots, A_{r+1})Z &= \\
&= (1 - \kappa) \left\{ A_{r+1}p^{-1}R(X, Y; A_1, \dots, A_r)Z - R(A_{r+1}p^{-1}X, Y; A_1, \dots, A_r)Z \right. \\
&\quad - R(X, A_{r+1}p^{-1}Y; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)(A_{r+1}p^{-1}Z) \\
&\quad \left. - \sum_{i=1}^r R(X, Y; A_1, \dots, A_{r+1}p^{-1}A_i, \dots, A_r)Z \right\} \\
&\quad + \frac{\kappa}{2} \left\{ p \left[ p^{-1}A_{r+1}, p^{-1}R(X, Y; A_1, \dots, A_r)Z \right] \right. \\
&\quad - R(p[p^{-1}A_{r+1}, p^{-1}X], Y; A_1, \dots, A_r)Z \\
&\quad - R(X, p[p^{-1}A_{r+1}, p^{-1}Y]; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)(p[p^{-1}A_{r+1}, p^{-1}Z]) \\
&\quad \left. - \sum_{i=1}^r R(X, Y; A_1, \dots, p[p^{-1}A_{r+1}, p^{-1}A_i], \dots, A_r)Z \right\}.
\end{aligned} \tag{69}$$

Now we can reverse the claim and using the exactly the same argument starting with the third equality in (65) we can prove that  $R(X, Y; A_1, \dots, A_r)Z$  is the linear combination of terms  $Sp$ , where  $S$  is some word which is a product of the terms  $Xp^{-1}, Yp^{-1}, A_1p^{-1}, \dots, A_r^{-1}$  of the first order. Similarly we end up with

$$\begin{aligned}
R(X, Y; A_1, \dots, A_{r+1})Z &= \\
&= (1 - \kappa) \left\{ R(X, Y; A_1, \dots, A_r)Zp^{-1}A_{r+1} - R(Xp^{-1}A_{r+1}, Y; A_1, \dots, A_r)Z \right. \\
&\quad - R(X, Yp^{-1}A_{r+1}; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)(Zp^{-1}A_{r+1}) \\
&\quad \left. - \sum_{i=1}^r R(X, Y; A_1, \dots, A_i p^{-1}A_{r+1}, \dots, A_r)Z \right\} \\
&\quad + \frac{\kappa}{2} \left\{ \left[ R(X, Y; A_1, \dots, A_r)Zp^{-1}, A_{r+1}p^{-1} \right] p \right. \\
&\quad - R([Xp^{-1}, A_{r+1}p^{-1}]p, Y; A_1, \dots, A_r)Z \\
&\quad - R(X, [Yp^{-1}, A_{r+1}p^{-1}]p; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)([Zp^{-1}, A_{r+1}p^{-1}]p) \\
&\quad \left. - \sum_{i=1}^r R(X, Y; A_1, \dots, [A_i p^{-1}, A_{r+1}p^{-1}]p, \dots, A_r)Z \right\}.
\end{aligned} \tag{70}$$

Now subtracting (70) from (69) and using the fact that

$$[Ap^{-1}, Bp^{-1}]p = p[p^{-1}A, p^{-1}B] = -p[p^{-1}B, p^{-1}A]$$



for any  $A, B \in H(n, \mathbb{C})$  and linearity of  $R(X, Y; A_1, \dots, A_r)Z$ , we get that

$$\begin{aligned}
& R(X, Y; A_1, \dots, A_{r+1})Z - R(X, Y; A_1, \dots, A_r)Z = 0 = \\
& = p \left[ p^{-1}A_{r+1}, p^{-1}R(X, Y; A_1, \dots, A_r)Z \right] - R(p[p^{-1}A_{r+1}, p^{-1}X], Y; A_1, \dots, A_r)Z \\
& - R(X, p[p^{-1}A_{r+1}, p^{-1}Y]; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)(p[p^{-1}A_{r+1}, p^{-1}Z]) \\
& - \sum_{i=1}^r R(X, Y; A_1, \dots, p[p^{-1}A_{r+1}, p^{-1}A_i], \dots, A_r)Z.
\end{aligned} \tag{71}$$

So in particular (69) is just

$$\begin{aligned}
& R(X, Y; A_1, \dots, A_{r+1})Z = \\
& = (1 - \kappa) \left\{ A_{r+1}p^{-1}R(X, Y; A_1, \dots, A_r)Z - R(A_{r+1}p^{-1}X, Y; A_1, \dots, A_r)Z \right. \\
& - R(X, A_{r+1}p^{-1}Y; A_1, \dots, A_r)Z - R(X, Y; A_1, \dots, A_r)(A_{r+1}p^{-1}Z) \\
& \left. - \sum_{i=1}^r R(X, Y; A_1, \dots, A_{r+1}p^{-1}A_i, \dots, A_r)Z \right\} \\
& = (1 - \kappa)p \left\{ p^{-1}A_{r+1}p^{-1}R(X, Y; A_1, \dots, A_r)Z - p^{-1}R(A_{r+1}p^{-1}X, Y; A_1, \dots, A_r)Z \right. \\
& - p^{-1}R(X, A_{r+1}p^{-1}Y; A_1, \dots, A_r)Z - p^{-1}R(X, Y; A_1, \dots, A_r)(A_{r+1}p^{-1}Z) \\
& \left. - \sum_{i=1}^r p^{-1}R(X, Y; A_1, \dots, A_{r+1}p^{-1}A_i, \dots, A_r)Z \right\}.
\end{aligned}$$

Considering again (64) and the first claim we get that the above is equivalent to

$$R(X, Y; A_1, \dots, A_{r+1})Z = (1 - \kappa)D(R(X, Y; A_1, \dots, A_r))[p][A_{r+1}],$$

which is just (63).  $\square$

Now again the Lie algebra  $\hat{\mathfrak{h}}_p(\nabla)$  is generated by the endomorphisms  $\nabla^r R$ . This means that the generated algebra grows as  $r$  increases and after some finitely many steps it stabilizes and taking higher covariant derivatives of  $R$  is unnecessary. Since the manifold  $P(n, \mathbb{C})$  is simply connected the holonomy group and the restricted holonomy group coincide, so  $\hat{\mathfrak{h}}_p(\nabla) = \mathfrak{h}_p(\nabla)$ . By the second formula in (65) and (63) we have the following

**Corollary 3** *The Lie algebra  $\mathfrak{h}_p(\nabla)$  is faithfully represented over the vector space  $V = H(n, \mathbb{C})$  (or  $V = H(n, \mathbb{R})$ ) with  $\rho : \mathfrak{h}_p(\nabla) \mapsto \text{End}(V)$  given as*

$$\rho(W)Z = ZW + W^*Z \tag{72}$$

for  $W \in \mathfrak{h}_p(\nabla)$  and  $Z \in H(n, \mathbb{C})$  (or  $Z \in H(n, \mathbb{R})$ ).

We are in position to do a case by case analysis for different values of  $\kappa$ .  $\mathfrak{so}(n, \mathbb{R})$  denotes the Lie algebra of skew-symmetric  $n$ -by- $n$  matrices over the real field  $\mathbb{R}$ ,  $\mathfrak{su}(n, \mathbb{C})$  denotes the Lie algebra of skew-hermitian matrices with vanishing trace over  $\mathbb{C}$ ,  $\mathfrak{sl}(n, \mathbb{F})$  denotes the Lie algebra of traceless matrices over the field  $\mathbb{F}$ .

**Theorem 10** *Let the smooth connected manifold  $P(n, \mathbb{C})$  with tangent space  $H(n, \mathbb{C})$  be equipped with real analytic connection*

$$\nabla_{X_p} Y_p = DY[p][X_p] - \frac{\kappa}{2} (X_p p^{-1} Y_p + Y_p p^{-1} X_p) \quad (73)$$

with  $\kappa \in \mathbb{R}$ . Then the holonomy algebra  $\mathfrak{h}_p(\nabla)$  is as follows:

$$\mathfrak{h}_p(\nabla) = \begin{cases} \text{the trivial algebra} & \text{if } \kappa = 0, 2, \\ \mathfrak{su}(n, \mathbb{C}) & \text{if } \kappa = 1, \\ \mathfrak{sl}(n, \mathbb{C}) & \text{else.} \end{cases} \quad (74)$$

In the case of the submanifold  $P(n, \mathbb{R})$  with tangent space  $H(n, \mathbb{R})$  we have

$$\mathfrak{h}_p(\nabla) = \begin{cases} \text{the trivial algebra} & \text{if } \kappa = 0, 2, \\ \mathfrak{so}(n, \mathbb{R}) & \text{if } \kappa = 1, \\ \mathfrak{sl}(n, \mathbb{R}) & \text{else.} \end{cases} \quad (75)$$

*Proof* By the conjugate invariance of  $\mathcal{H}_p(\nabla)$  it is enough to consider the case when  $p = I$ .

Suppose  $\kappa = 0, 2$ . Then the curvature (62) of the connection vanishes, so  $\mathfrak{h}_p(\nabla)$  is the trivial algebra.

Suppose  $\kappa = 1$ . Then the curvature (62) is nonzero, but is covariantly constant, all first and higher order covariant derivatives vanish due to Theorem 9. Therefore the manifold is a symmetric space that is very well known and the algebra  $\mathfrak{h}_p(\nabla)$  by (65) is generated by elements of the form  $[X, Y]$  where  $X, Y \in H(n, \mathbb{F})$ . We have for all  $[X, Y] = W \in \mathfrak{h}_p(\nabla)$  that

$$W^* = [X, Y]^* = -[X^*, Y^*] = -[X, Y]$$

where  $*$  can be replaced by the transpose  $^T$  over  $\mathbb{F} = \mathbb{R}$ . Also since  $\text{Tr} W = \text{Tr}[X, Y] = 0$  we have  $\mathfrak{h}_p(\nabla) = \mathfrak{so}(n, \mathbb{R})$  if  $\mathbb{F} = \mathbb{R}$  and  $\mathfrak{h}_p(\nabla) = \mathfrak{su}(n, \mathbb{C})$  if  $\mathbb{F} = \mathbb{C}$ .

Suppose  $\kappa \neq 0, 1, 2$ . Then by Theorem 9 the higher order covariant derivatives  $\nabla^r R$  (as we will see immediately) no longer vanish. Let  $W \in \mathfrak{h}_p(\nabla)$ . Then by (65), (63) and Corollary 3

$$W = \left( \frac{\kappa}{2} - \frac{\kappa^2}{4} \right) (1 - \kappa)^r D([p^{-1} X, p^{-1} Y])[p][A_1, \dots, A_r],$$

where  $X, Y \in H(n, \mathbb{F})$ . I.e.  $W$  is given by the linear combination of commutators of some  $n$ -by- $n$  matrices over the field  $\mathbb{F}$ , so  $\text{Tr} W = 0$ . This tells us that

$$\mathfrak{h}_p(\nabla) \subseteq \mathfrak{sl}(n, \mathbb{F}). \quad (76)$$

Now we will show that the generated algebra already stabilizes for  $r = 1$ . Without loss of generality we can assume that  $p = I$ . Then we have to consider the generators of the form

$$G = D([p^{-1} X, p^{-1} Y])[I][A_1] = -[A_1 X, Y] - [X, A_1 Y]. \quad (77)$$

Let

$$E_{ik}^+ = \begin{cases} E_{ik} + E_{ki} & \text{if } i \neq k, \\ E_{ii} & \text{if } i = k, \end{cases}$$

$$E_{ik}^- = E_{ik} - E_{ki}$$

where  $E_{ik}$  is the matrix with zero entries excluding the  $(ik)$  entry which is 1. Then  $E_{ik}^+$  form a basis of  $H(n, \mathbb{R})$  and  $E_{ik}^-$  form a basis of the vector space of skew-hermitian matrices  $SH(n, \mathbb{R})$  over the real field  $\mathbb{R}$ . The vector space  $SH(n, \mathbb{C})$  is defined similarly over  $\mathbb{C}$ . Note also that  $H(n, \mathbb{C}) \cong H(n, \mathbb{R}) \oplus SH(n, \mathbb{R})$  and that  $E_{ik}^+ E_{lm}^- = 0$  in general. Suppose that  $A_1 = E_{iz}^+$ ,  $X = E_{ky}^+$  and  $Y = E_{ik}^+$ . Then by (77)

$$G = -E_{iz}^+ E_{ky}^+ E_{ik}^+ + E_{ik}^+ E_{iz}^+ E_{ky}^+ - E_{ky}^+ E_{iz}^+ E_{ik}^+ + E_{iz}^+ E_{ik}^+ E_{ky}^+.$$

Using that  $E_{ik}^+ = E_{ki}^+$  and imposing restrictions  $z \neq k$  and  $y \neq i$  we get that

$$G = \begin{cases} E_{zy} & \text{if } z \neq y, \\ E_{zz} - E_{ii} & \text{if } y = z. \end{cases}$$

So the matrices  $G$  of this form span the whole  $\mathfrak{sl}(n, \mathbb{R})$ , i.e. considering (76) we have  $\mathfrak{h}_p(\nabla) = \mathfrak{sl}(n, \mathbb{R})$  if  $\mathbb{F} = \mathbb{R}$ . Similarly if  $A_1 = E_{iz}^-$ ,  $X = E_{ky}^-$  and  $Y = E_{ik}^-$ , then we get the same generator  $G$ , so  $\mathfrak{h}_p(\nabla) = \mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R})$  due to  $E_{ik}^+ E_{lm}^- = 0$  if  $\mathbb{F} = \mathbb{C}$ ; that is  $\mathfrak{h}_p(\nabla) = \mathfrak{sl}(n, \mathbb{C})$  in the complex case.  $\square$

By the proof of the previous Theorem 10 we see that  $R_{jkl;m}^s = 0$  everywhere if and only if  $\kappa = 0, 1, 2$ . This proves the following

**Corollary 4** *The only matrix means which are affine means corresponding to symmetric spaces are the arithmetic, harmonic and geometric means.*

Since we know the holonomy groups we can decide their metrizability.

**Corollary 5** *The affine connections (73) are metric in the following cases:*

1.  $n = 1, 2$ ,  $\kappa$  arbitrary,  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ,
2.  $n \geq 3$ ,  $\kappa = 0, 1, 2$ ,  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ .

*Proof* The case  $n = 1$  is trivial. In [27] all irreducible holonomies of affine connections are classified and metrizability is also discussed. The metric connections were classified by Berger long ago. The holonomy  $\mathfrak{sl}(2, \mathbb{R})$  is isomorphic to  $\mathfrak{sp}(2, \mathbb{R})$  which is metric, there exists an invariant symplectic form. Also  $\mathfrak{sl}(2, \mathbb{C})$  is isomorphic to  $\mathfrak{so}(1, 3)$  which is also metric, there exists an invariant pseudo-Riemannian metric with signature  $(1, 3)$ . This isomorphic correspondence fails in higher dimensions  $n \geq 3$ , where the holonomies  $\mathfrak{sl}(n, \mathbb{F})$  ( $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ ) with representation over  $H(n, \mathbb{F})$  is non-metric.  $\square$

*Remark 4* In the second case in Corollary 5 although there exists no metric structure, however by inspecting the holonomy group  $\mathcal{H}_p(\nabla)$  we get that there exist totally geodesic flat submanifolds. That is if we consider the subset  $D(n, \mathbb{F})$  of diagonal matrices of  $H(n, \mathbb{F})$  in both cases  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ , we get a totally geodesic Euclidean submanifold and a Riemannian metric on  $D(n, \mathbb{F})$  is given in the form

$$\text{Tr} \left\{ p^{-2\kappa} \log_p^2(q) \right\}$$

where  $\log_p(q)$  is the logarithm map given in (37).

So there exist no previously unknown affine matrix mean which correspond to a Riemannian manifold. Although we have found a previously unknown, generally non-metrizable, one parameter family of affinely connected manifolds where the points of the geodesics are matrix means, in particular matrix power means.

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